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The Mathematics of Harmony and "Golden" Non-Euclidean Geometry as the "Golden" Paradigm of Modern Science, Geometry, and Computer Science

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Abstract

In 1962, American physicist and historian of science Thomas Kuhn (1922-1996) published the book "The Structure of Scientific Revolutions". In this book, Kuhn had substantiated a concept of changing "scientific paradigms" as premise of the "scientific revolutions." What was the major idea (paradigm) of the ancient Greek science? A majority of researchers give the following answer: the idea of Harmony associated with the "golden ratio" and "Platonic Solids." The prominent Russian philosopher Alexey Losiv argues: "*From Plato's point of view, and in general in terms of the entire ancient cosmology, the Universe is determined as a certain proportional whole that obeys the law of harmonic division - the golden section.*" That is, Losev put the "golden ratio" in the center of the "golden" paradigm of ancient Greek science. Stakhov's book "The Mathematics of Harmony" (World Scientific, 2009) and Stakhov & Aranson's book "The "Golden" Non-Euclidean Geometry. (World Scientific, 2016) can be seen as the first attempts to revive in modern science and mathematics the "golden" paradigm of ancient Greek science. The main goal is to show that the modern Mathematics of Harmony contains a large number of excellent mathematical results, which can decorate many of the traditional mathematical disciplines. We are talking about new look on the history of mathematics (Proclus' hypothesis), about new classes of recurrent relations and new mathematical constants, following from Pascal's triangle, about new types of recursive hyperbolic functions, about the original solution of Hilbert's Fourth Problem, which underlies the "Golden" non-Euclidean geometry, about new numeral systems with irrational bases for computer science. In addition, the Mathematics of Harmony develops new original view on special theory of relativity, which leads us to a new view on the fundamental physical constants such as the Fine-Structure Constant.

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Introduction

***Alexey Losev and Johannes Kepler's quotes about the "golden ratio**.*" What was the major idea of the ancient Greek science? A majority of researchers give the following answer: the idea of Harmony associated with the "golden ratio" and "Platonic Solids." As it is known, in ancient Greek philosophy, Harmony was in opposition to Chaos and meant self-organization of the Universe. Alexey Losev, the famous Russian philosopher of the aesthetics of antiquity and the Renaissance, evaluates the main achievements of the ancient Greeks in this area as follows [1]:

"*From Plato's point of view, and in general in terms of the entire ancient cosmology, the Universe is determined as a certain proportional whole that obeys the law of harmonic division - the golden ratio .... The ancient Greek system of cosmic proportions in the literature is often interpreted as a curious result of unrestrained and wild imagination. In such explanations we can see the unscientific helplessness of those who claim it. However, we can understand this historical and aesthetic phenomenon only in connection with a holistic understanding of history, that is, by using a dialectical view of culture and looking for the answer in the peculiarities of ancient social life.*"

Here Losev formulates the "*golden" paradigm of ancient cosmology.* It is based upon the most important ideas of ancient science that are sometimes treated in modern science as a "*curious result of an unrestrained and wild imagination.*" First of all, we are talking about the Pythagorean Doctrine of the Numerical Harmony of the Universe and Plato's cosmology, based on the Platonic solids. Referring to the geometrical structure of the Cosmos and its mathematical relations, expressing Cosmic Harmony, the Pythagoreans had anticipated the modern mathematical basis of the natural sciences, which began to develop rapidly in the 20th century. Pythagoras's and Plato's idea about Cosmic Harmony proved to be immortal.

Johannes Kepler, brilliant astronomer and author of "Kepler's laws," expressed his admiration for the golden ratio in the following words:

"*Geometry has two great treasures: one is the theorem of Pythagoras; the other the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel.*" (taken from [2]). You should recall that the ancient problem of the division of a line segment in extreme and mean ratio was Euclid's language for the golden ratio!

Thus, the idea of Harmony, which underlies the ancient Greek doctrine of Nature, was the main "paradigm" of Greek science starting from Pythagoras and ending by Euclid. This relates directly to the golden ratio and Platonic solids, which are the most important Greek mathematical discoveries, which had expressed Universe Harmony.

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Pythagoreanism and Pythagorean MATHEM's. By studying sources of mathematics origins, we inevitably come to *Pythagoras* and his doctrine, called Pythagoreanism [3]. According to the tradition, Pythagoreans were divided into two separate schools of thought, the *mathēmatikoi (mathematicians)* and the *akousmatikoi (listeners)*. Listeners developed religious and ritual aspects of *Pythagoreanism, mathematicians* studied four Pythagorean MATHEM's: *arithmetic, geometry, harmonics and spherics*. These MATHEM's, according to Pythagoras, were the main component parts of mathematics. Unfortunately, the Pythagorean MATHEM of *harmonics* was lost in mathematics in the process of historical development.

Applications of the Golden Ratio and Metallic Means in modern science. In recent years, there is seen an important trend in the development of modern science. Its essence lies in the revival of "harmonic ideas" of Pythagoras, Plato and Euclid, based on the "golden ratio" [4, 5], and the so-called "silver and metallic proportions" [6.7], in different areas of modern science: history and foundations of mathematics [8-22], geometry [23-31], number theory, measurement theory and new numeral systems [32-48], new theory of hyperbolic functions [41-45], matrix theory and coding theory [46-48], millennium problems [40,50], modern metaphysics [51], genetics [52], crystallography [53-55], chemistry [56], botany [24], and finally, philosophy [57, 58] and other important areas of modern science.

The Mathematics of Harmony and "Golden" Non-Euclidean Geometry. Stakhov's book "The Mathematics of Harmony. From Euclid to Contemporary Mathematics and Computer Science" (World Scientific, 2009) [2] and Stakhov & Aranson's book "Golden" Non-Euclidean Geometry. Hilbert's Fourth Problem, "Golden" Dynamical System, and Fine-Structure Constant" (World Scientific, 2016) [27] can be seen as the first attempts to revive in modern science and mathematics, the Pythagorean MATEM of "harmonics" and also the "harmonic ideas" of Pythagoras, Plato and Euclid. It is shown in the books [2,6,7,23,26,27,33,34,51,57,58] and the articles [8 - 22,25,28-32, 35-50,52-56] that the "harmonic ideas" of Pythagoras, Plato and Euclid affect not only the foundations of modern theoretical physics (in particular, *Einstein's special theory of relativity*) and the foundations of mathematics (in particular, *number theory and geometry*), but they can also be source of the *numeral systems with irrational bases*, which are the basis of new computer projects (Fibonacci computers).

Thomas Kuhn's "The Structure of Scientific Revolutions." The above-mentioned publications, demonstrating effective applications of the "golden ratio" and "metallic proportions" in various fields of modern science, are convincing confirmation of the fact that the "golden" scientific revolution is brewing in modern science [59]. In 1962, American physicist and historian of science Thomas Kuhn (1922-1996) published the book "The Structure of Scientific Revolutions" [60]. In this book, Kuhn had substantiated a concept of changing "scientific paradigms" as premise of the "scientific revolutions." According to Kuhn, the paradigm is a scientific concept, which unites the members of scientific community and, conversely, a scientific community consists of people who recognize a certain paradigm. As a rule, the paradigm is reflected in books, the works of scholars and for many years determines the range of problems and methods of their solution in a particular area of science. As examples of such paradigms, Kuhn points out on Aristotle's concepts, Newtonian mechanics, and so on.

As it is known, the notion of "paradigm" derives from the Greek "paradeigma" (the example or pattern) and means the combination of explicit and implicit (and often not perceived) prerequisites, which determine the directions scientific researches, recognized at the given stage of science development.

The concept of "paradigm," in the modern sense, had introduced by Thomas Kuhn. Changing paradigms ("paradigm shift") is a notion, also first introduced by Thomas Kuhn [60] to describe the changes in the basic assumptions within the leading theory of science (paradigm).

Usually changing scientific paradigm is among the most dramatic events in the history of science. When a scientific discipline is changing one paradigm to another, this is called "scientific revolution" or "paradigm shift," according to Kuhn's terminology, The decision to abandon the old paradigm is always at the same time a decision to take new paradigm, and the decision to change paradigm includes a comparison of both paradigms with nature phenomena ("Authority of Nature" [61]) and the comparison of paradigms with each other.

What is the "golden" paradigm? To answer this question, we turn once again to the well-known quote of the Russian genius of philosophy, researcher of aesthetics of ancient Greece and the Renaissance Alexey Losev (1893-1988). In this quote, Losev in very clear form had formulated the essence of the "golden" paradigm of ancient cosmology. The most important ideas of ancient science, which in modern science are sometimes treated as a "curious result of unrestrained and wild imagination," underlie the "golden" paradigm of ancient science. First of all, these are "the Pythagoras Doctrine of Numerical Harmony of the Universe" and "Plato's Cosmology," based on the Platonic solids. Thus Losev argues:

"***From Plato's point of view, and in general in terms of the entire ancient cosmology, the Universe is determined as a certain proportional whole that obeys the law of harmonic division - the golden section.***"

That is, Losev put the "golden ratio" in the center of the "golden" paradigm of ancient science. Thus, referring to the geometrical structure of the Universe and to the geometric relationships, expressing harmony, in particular, to the "golden ratio", the Pythagoreans anticipated the emergence of mathematical natural sciences, which began to develop rapidly in the 20th century. The idea of Pythagoras and Plato about the universal harmony of the Universe proved to be immortal.

Proclus' hypothesis: a New View on Euclid's Elements and the History of Mathematics. Platonic solids had played a special role in Euclid's *Elements*. Their theory is described by Euclid in the 13th, that is, the final book of his *Elements*.

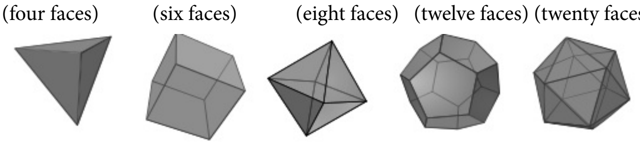

Figure 1: Platonic Solids.

The question arises, what purpose pursued Euclid by placing Platonic solids in the final book of his Elements. Proclus' hypothesis, formulated in the 5th century by the Greek philosopher and mathematician

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Proclus Diadochus (412 – 485), contains a surprising answer to this question [22].

According to Proclus, Euclid's main goal at writing his *Elements* was not simply to set forth geometry itself what is traditional view on Euclid's *Elements*, but to build a complete theory of the regular polyhedra ("Platonic solids") what was an essential element (paradigm) of Plato's cosmology. This theory was outlined by Euclid in the concluding or thirteenth book of the *Elements*. This fact in itself is an indirect confirmation of Proclus' hypothesis. Often the most important scientific information is found in the concluding part of scientific publication.

Proclus' hypothesis is the most unexpected hypothesis in the history of mathematics, which alters our traditional ideas about the history of mathematics. It follows from this hypothesis that Euclid's *Elements* are the source of the two directions in the history of mathematics - *Classical Mathematics* and *Mathematics of Harmony*.

For many centuries, the creation of Classic Mathematics, Queen of the Natural Sciences, was the main goal of mathematicians. However, by studying the works of Pythagoras, Plato, Euclid, Fibonacci, Pacioli, and Kepler, we can conclude that the intellectual forces of many prominent mathematicians and thinkers were directed towards the development of the basic concepts and applications of the Mathematics of Harmony what, according to Proclus, was the main "paradigm" of Euclid's Elements. Unfortunately, these two important mathematical approaches (*Classic Mathematics* and the *Mathematics of Harmony*) evolved separately from one another. The time has come to unite these two mathematical approaches. This important unification can now lead to novel scientific discoveries in both mathematics and the theoretical natural sciences.

The main goal of the present article. The main goal is to show that the modern Mathematics of Harmony, which revives the Pythagorean MATEM of "harmonics," contains a large number of excellent mathematical results, which can decorate many of the traditional mathematical disciplines. We are talking about new look on the history of mathematics (Proclus' hypothesis [22]), about new classes of recurrent relations, which expand a theory of Fibonacci numbers, about new mathematical constants, following from Pascal's triangle, about new types of hyperbolic functions [41-45], about the original solution of Hilbert's Fourth Problem [21,49], which underlies the "Golden" non-Euclidean geometry [27], about new numeral systems for computer science [39]. In addition, in the Mathematics of Harmony we develop the original look on special theory of relativity, which leads us to a new view of the fundamental physical constants such as the Fine-Structure Constant [50].

The Golden Ratio, Fibonacci and Lucas numbers

The Golden Ratio. The Golden Ratio

$$\Phi=\frac{1+\sqrt{5}}{2} \quad (1)$$

is a positive root of the following algebraic equation:

$$x^2=x+1 \quad (2)$$

By using (1), (2), we get the following identities for the golden ratio:

$$\Phi^2=\Phi+1 \quad (3)$$

$$\Phi=1+\frac{1}{\Phi} \quad (4)$$

$$\Phi^n=\Phi^{n-1}+\Phi^{n-2},\ n=0,\pm1,\pm2,\pm3,\ldots \quad (5)$$

$$\Phi=\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}} \quad (6)$$

$$\Phi=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}} \quad (7)$$

The expression (7) has a profound mathematical sense. The Russian mathematicians A.Y. Khinchin [62] and N.N. Vorobyov [63] drew attention to the fact that the continued fraction (7) singles out the golden ratio amongst all irrationals as unique irrational number.

The Golden Ratio in the Dodecahedron and Icosahedron. The dodecahedron (Figure1d) and its dual the icosahedron (Figure 1e) play a special role among the Platonic solids. First of all, we emphasize that the geometry of the dodecahedron and the icosahedron relate directly to the golden ratio. Indeed, all faces of the dodecahedron are regular pentagons, based on the golden ratio. If we look closely at the icosahedron in Figure 1e, we can see that in each of its vertices the five triangles come together and their outer edges form a regular pentagon. These facts reveal the crucial role the golden ratio in the geometric construction of these two Platonic solids.

However, there are other deep mathematical confirmations of the fundamental role of the golden ratio in the icosahedron and dodecahedron. The Platonic solids have three specific spheres. The first (*inner*) sphere is inscribed inside the Platonic solid and touches the centers of its faces. We denote the radius of the inner sphere as R_i. The second or *middle* sphere of the Platonic solid touches its ribs. We denote the radius of the middle sphere as R_m. Finally, the third (*outer*) sphere is circumscribed around the Platonic solid and passes through its vertices. We denote its radius as . In geometry, it is demonstrated that the values of the radii of these spheres for the dodecahedron and the icosahedron with an edge of unit length is expressed through the golden ratio (Table 1).

	R_c	R_m	R_i
Icosahedron	$\frac{1}{2}\Phi\sqrt{3-\Phi}$	$\frac{1}{2}\Phi$	$\frac{\frac{1}{2}\Phi^2}{\sqrt{3}}$
Dodecahedron	$\frac{\Phi\sqrt{3}}{2}$	$\frac{\Phi^2}{2}$	$\frac{\Phi^2}{2\sqrt{3-\Phi}}$

Table 1: The Golden Ratio in the spheres of the dodecahedron and icosahedron.

Note that the ratio of the radii $\frac{R_c}{R_i}=\frac{\sqrt{3(3-\Phi)}}{\Phi}$ is the same for both the icosahedron and dodecahedron. Thus, if the dodecahedron and icosahedron have the same inner spheres, their outer spheres are equal as well. This is a reflection of the hidden harmony in both the dodecahedron and icosahedron.

Fibonacci and Lucas numbers. Let us consider the following recurrent relations, together with the initial terms (seeds):

$$F_n=F_{n-1}+F_{n-2};\ F_1=1,F_2=1 \quad (8)$$

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$$L_n = L_{n-1} + L_{n-2};\ L_1 = 1, L_2 = 3$$

These recurrent relations for the given initial terms (seeds) generate two well-known numerical sequences:

Fibonacci numbers: F_n 1,1,2,3,5,8,13,21,34,55,89, … (10)

Lucas numbers: L_n 1,3,4,7,11,18,29,47,76,123, … (11)

Fibonacci and Lucas numbers can be extended toward the side of the negative values of the indeces n. Table 2 shows examples of the "extended" Fibonacci and Lucas numbers.

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
F_{-n}	0	1	−1	2	−3	5	−8	13	−21	34	−55
L_n	2	1	3	4	7	11	18	29	47	76	123
L_{-n}	2	−1	3	−4	7	−11	18	−29	47	−76	123

Table 2: The "extended" Fibonacci ad Lucas numbers.

The "extended" Fibonacci and Lucas numbers are connected by the following simple relations:

$$F_{-n} = (-1)^{n+1} F_n \quad (12)$$

$$L_{-n} = (-1)^n L_n \quad (13)$$

Fibonacci Q-matrix. In recent decades the theory of Fibonacci numbers has been supplemented by the theory of the so-called Fibonacci Q-matrix [64]. This is a (2×2)-matrix of the following form:

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \quad (14)$$

The Fibonacci Q-matrix has the following remarkable properties:

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \quad (15)$$

$$\det Q = -1,\ \det Q^n = F_{n-1}F_{n+1} - F_n^2 = (-1)^n \quad (16)$$

n	0	1	2	3	4	5
Q^n	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$	$\begin{pmatrix} 8 & 5 \\ 5 & 3 \end{pmatrix}$
Q^{-n}	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$	$\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$	$\begin{pmatrix} -3 & 5 \\ 5 & -8 \end{pmatrix}$

(17)

Cassini's formula for Fibonacci numbers. The history of science does not reveal why French astronomer Giovanni Domenico Cassini (1625-1712) took such a great interest in Fibonacci numbers. Most likely it was simply an object of rapture for the great astronomer. At that time many serious scientists were fascinated by Fibonacci numbers and the golden ratio. We recall that these numbers were also an aesthetic object for Johannes Kepler, who was a contemporary of Cassini.

Cassini's contribution into Fibonacci numbers theory is a proof of the following formula, named Cassini's formula:

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1} \quad (18)$$

Connection of Fibonacci numbers with the golden ratio. By using (7), we can represent fractional approximation of the golden ratio as follows:

$1 = \frac{1}{1}$ (first approximation);

$1 + \frac{1}{1} = \frac{2}{1}$ (second approximation);

$1 + \frac{1}{1 + \frac{1}{1}} = \frac{3}{2}$ (third approximation);

$1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}} = \frac{5}{3}$ (fourth approximation).

By continuing this process, we find the sequence of continued fractions approximating the golden ratio, which is the sequence of the ratios of adjacent Fibonacci numbers. That is,

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \ldots \to \Phi = \lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2} \quad (19)$$

The sequence (19) expresses the famous law of phyllotaxis [24], according to which Nature constructs pine cones, pineapples, cacti, heads of sunflowers and other botanical objects. In other words, Nature uses this unique mathematical property of the golden ratio (19) and (1.20) in its wonderful creations!

Binet's Formulas and Recursive Hyperbolic Fibonacci and Lucas functions

Binet's formulas for Fibonacci and Lucas numbers. French mathematician of 19th century Jacques Philippe Marie Binet (1776-1856) is well-known in mathematics due his now famous *Binet's formulas*. These formulas link the "extended" Fibonacci and Lucas numbers directly with the golden ratio and without a doubt are amongst the most famous mathematical formulae:

$$F_n = \begin{cases} \dfrac{\Phi^n + \Phi^{-n}}{\sqrt{5}} & \text{for } n = 2k+1 \\ \dfrac{\Phi^n - \Phi^{-n}}{\sqrt{5}} & \text{for } n = 2k, \end{cases} \quad (20)$$

$$L_n = \begin{cases} \Phi^n + \Phi^{-n} & \text{for } n = 2k \\ \Phi^n - \Phi^{-n} & \text{for } n = 2k+1. \end{cases} \quad (21)$$

We know that the "extended" Fibonacci and Lucas numbers are always integers. But any degree of the golden ratio is an irrational number. It follows from this that the integer numbers and can be represented by using Binet's formulas (20) and (21) through the irrational number, the golden ratio !

Classic hyperbolic functions. The functions:

$$sh(x) = \frac{e^x - e^{-x}}{2},\ ch(x) = \frac{e^x + e^{-x}}{2} \quad (22)$$

are called, respectively, hyperbolic sine and hyperbolic cosine.

These analytical definitions (22) can be used to obtain some very important identities of hyperbolic trigonometry, in particular, the parity property:

$$sh(-x) = -sh(x);\quad ch(-x) = ch(x);\quad th(-x) = th(x) \quad (23)$$

and the analog of Pythagoras theorem:

$$ch^2x - sh^2x = 1$$

The interest in hyperbolic functions (22) increased significantly during the 19th century, when the Russian geometer Nikolay Lobachevsky (1792 - 1856) used them to describe mathematical relationships for non-Euclidian geometry. It was for this reason, Lobachevsky's geometry became to be known as *hyperbolic geometry*.

The recursive hyperbolic Fibonacci and Lucas functions. By comparing the hyperbolic functions (22) to Binet's formulas (20), (21), we can see that they are similar by their structure. This simple observation led to the introduction of a new class of hyperbolic functions, the *recursive hyperbolic Fibonacci and Lucas functions* [41]. These fnctions are a very significant step in the development of a general theory of hyperbolic functions and hyperbolic geometry. New hyperbolic functions look as follows:

Recursive hyperbolic Fibonacci sine

$$sF(x) = \frac{\Phi^x - \Phi^{-x}}{\sqrt{5}} \tag{25}$$

Recursive hyperbolic Fibonacci cosine

$$cF(x) = \frac{\Phi^x + \Phi^{-x}}{\sqrt{5}} \tag{26}$$

Recursive hyperbolic Lucas sine

$$sL(x) = \Phi^x - \Phi^{-x} \tag{27}$$

Recursive hyperbolic Lucas cosine

$$cL(x) = \Phi^x + \Phi^{-x} \tag{28}$$

The following simple relations connect the hyperbolic Fibonacci functions (25) and (26) with the hyperbolic Lucas functions (27) and (28):

$$sF(x) = \frac{sL(x)}{\sqrt{5}}; \quad cF(x) = \frac{cL(x)}{\sqrt{5}}. \tag{29}$$

Hyperbolic and recursive properties of the hyperbolic Fibonacci and Lucas functions. Hyperbolic Fibonacci and Lucas functions (25) - (28) are analogs of the classic hyperbolic functions (22); on the other hand, they are derived from Binet's formulas (20), (21). Their main feature and uniqueness consists in the fact that they retain all properties of the classic hyperbolic functions (22) (*hyperbolic properties*) and fractal properties of Fibonacci and Lucas numbers (*recursive properties*), represented in the form of Binet's formulas (20), (21).

The *parity properties* are the first example of the *hyperbolic properties*, similar to the properties (23):

$$sF(-x) = -sF(x); \quad cF(-x) = cF(x) \tag{30}$$

$$sL(-x) = -sL(x); \quad cL(-x) = cL(x) \tag{31}$$

The *analogs of Pythagoras theorem for hyperbolic Fibonacci* and *Lucas functions*, similar to (24), are other example of the hyperbolic properties:

$$[cF(x)]^2 - [sF(x)]^2 = \frac{4}{5} \tag{32}$$

$$[cL(x)]^2 - [sL(x)]^2 = 4 \tag{33}$$

Other hyperbolic properties are derived in the article [41].

The *recursive properties* of the hyperbolic Fibonacci and Lucas functions (25) - (28) are another confirmation of the unique nature of this new class of hyperbolic functions (25)-(28), because the classic hyperbolic functions (22) do not have similar properties. Let us now consider some recursive properties of the hyperbolic Fibonacci and Lucas functions (25) - (28) in comparison to similar properties of the "extended" Fibonacci and Lucas numbers represented as *Binet's formulas* (20), (21) .

Let us start from the simplest recurrent relation for Fibonacci numbers:

$$F_{n+2} = F_{n+1} + F_n. \tag{34}$$

It is proved in [41] that the fundamental recurrent relation (34) in terms of the hyperbolic Fibonacci functions (25), (26) looks as follows:

$$sF(x+2) = cF(x+1) + sF(x) \tag{35}$$

$$cF(x+2) = sF(x+1) + cF(x) \tag{36}$$

We have for the case of recurrent relation for Lucas numbers the similar relations:

$$L_{n+2} = L_{n+1} + L_n \tag{37}$$

$$sL(x+2) = cL(x+1) + sL(x) \tag{38}$$

$$cL(x+2) = sL(x+1) + cL(x) \tag{39}$$

There are the following relations [41], derived from Cassini's formula (18):

$$[sF(x)]^2 - cF(x+1)\,cF(x-1) = -1 \tag{40}$$

$$[cF(x)]^2 - sF(x+1)\,sF(x-1) = 1 \tag{41}$$

By comparing Binet's formulas (20), (21) with the hyperbolic Fibonacci and Lucas functions (25) - (28), it is easy to see that for the discrete values of the variable x (x=0,±1,±2,±3,…) the functions (25), (26) coincide with the "extended" Fibonacci numbers, calculated according to Binet's formula (20), that is,

$$F_n = \begin{cases} sF(n) & \text{for } n = 2k \\ cF(n) & \text{for } n = 2k+1 \end{cases} \tag{42}$$

and the functions (27), (28) coincide with the "extended" Lucas numbers, calculated according to Binet's formula (21), that is,

$$L_n = \begin{cases} cL(n) & \text{for } n = 2k \\ sL(n) & \text{for } n = 2k+1 \end{cases} \tag{43}$$

where k is any value from the set k=0,±1,±2,±3,… .

About new mathematical term of "recursive hyperbolic functions." The concepts of *gnomon, recursion, recursive function, and recurrent relation*, which reflect a fundamental property of nature - the property of *self-similarity* [65], are widespread in science, mathematics and computer science. The remarkable book Gnomon: *From Pharaohs to Fractals* by Midhat Gazale [7] is devoted to the study of the relationship between these concepts.

In Gazale's book [7] the concept of the "*gnomon*" is defined as follows: "*Gnomon is the figure, which, when added to another figure, forms a new figure, similar to the original.*"

Note that many objects in Nature are built according to this principle, which is called *self-similarity* [65]. A *self-similar* object coincides exactly or approximately with a part of itself (i.e. the whole has the same form as one or more of its parts). Many real-world objects, such as coastlines, have the property of *statistical self-similarity*: their parts are statistically homogeneous at different measurement scales. *Self-similarity* is a characteristic feature of *fractals*.

In mathematics, the principle of self-similarity is expressed by notions of *recursion* and *recursive function*. The Fibonacci and Lucas numerical sequences are the most striking examples of recursive functions in mathematics. They are given by recurrent formulas, where each member of the numerical sequence is calculated as a function of the n previous members. Thus, by using a finite expression

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(which is a combination of the recurrent formula and the set of values for the first terms of the series) we can obtain an infinite number of members of the sequence.

The unique nature of the new class of hyperbolic functions (25)-(28) is their fundamental connection to Fibonacci and Lucas numbers, which reflect the principle of *recursion, recursive functions* and ultimately the principle of *self-similarity*, the fundamental principle of Nature.

These considerations allow us to introduce a new mathematical concept of the *recursive hyperbolic functions* (25)-(28). The word "*recursive*" in this definition emphasizes the fundamental relationship of a new class of hyperbolic functions with the principle of *self-similarity*, the fundamental principle of Nature.

The "golden" Q-matrices. Let us represent the Fibonacci Q-matrix (15) $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ in the form of two matrices that are given for the even (n=2k) and odd (n=2k+1) values of n as follows:

$$Q^{2k} = \begin{pmatrix} F_{2k+1} & F_{2k} \\ F_{2k} & F_{2k-1} \end{pmatrix} Q^{2k+1} = \begin{pmatrix} F_{2k+2} & F_{2k+1} \\ F_{2k+1} & F_{2k} \end{pmatrix}. \qquad (44)$$

By using the relations (44) $F_n = \begin{cases} sF(n) \text{ for } n = 2k \\ cF(n) \text{ for } n = 2k+1 \end{cases}$, which connect Fibonacci numbers F_n with the hyperbolic Fibonacci functions (25), (26), we can represent the matrices (44) in terms of the hyperbolic Fibonacci functions (25), (26):

$$Q^{2k} = \begin{pmatrix} cF(2k+1) & sF(2k) \\ sF(2k) & cF(2k-1) \end{pmatrix}, Q^{2k+1} = \begin{pmatrix} sF(2k+2) & cF(2k+1) \\ cF(2k+1) & sF(2k) \end{pmatrix}, \qquad (45)$$

where k is a discrete variable, k=0,±1,±2,±3,….

If we exchange the discrete variable k in the matrices (45) by the continuous variable x, we obtain two unusual matrices that are functions of the continuous variable x:

$$Q_1(x) = \begin{pmatrix} cFs(2x+1) & sFs(2x) \\ sFs(2x) & cFs(2x-1) \end{pmatrix}, Q_2(x) = \begin{pmatrix} sFs(2x+2) & cFs(2x+1) \\ cFs(2x+1) & sFs(2x) \end{pmatrix}. \qquad (46)$$

The matrices (46) are called the golden Q-matrices [47]. They are a generalization of the Fibonacci Q-matrices for the continuous domain.

Taking into consideration the properties (40) and (41), we can write the following remarkable formulas for the determinants of the "golden" Q-matrices (46):

$$\det Q_1(x) = cFs(2x+1)cFs(2x-1) - \left[sFs(2x)\right]^2 = 1 \qquad (47)$$

$$\det Q_2(x) = cFs(2x+2)cFs(2x) - \left[sFs(2x+1)\right]^2 = -1. \qquad (48)$$

Bodnar's geometry: new hyperbolic geometry of phyllotaxis. The phenomenon of phyllotaxis [66] is one of the most wide-spread phenomenons of Nature. It is inherent for many biological systems. An essential feature of phyllotaxis consists in the spiral disposition of leaves on stems of plants and trees, petals in flower heads, seeds in pine cones and sunflower heads, etc. It is curious that Nature makes extensive use of Fibonacci numbers and other similar numerical sequences in phyllotaxis objects. This phenomenon, recognized since the time of Leonardo da Vinci, was a subject of discussion for many scientists, including Kepler, Weil and Turing, amongst others. It is intriguing that more complex concepts of symmetry are used in phyllotaxis, particularly *helical symmetry.*

In modern science, the greatest contribution into phyllotaxis theory was made by the Ukrainian architect Oleg Bodnar [18-20, 24].

We will not delve deeply into *Bodnar's geometry* which led to his new geometrical theory of phyllotaxis. Readers who want to become more acquainted with *Bodnar's geometry* are referred to his book [24]. Also we refer the readers to authors' article published in 2011 (in three parts) by the journal "Applied Mathematics" [18-20].

From *Bodnar's geometry* [18-20, 24] we can get the following important conclusions:

1. Bodnar's geometry has opened a new "hyperbolic world" for science – the world of phyllotaxis and its geometric secrets. The main feature of this world is that its basic mathematical relations are described by the recursive hyperbolic Fibonacci functions (25), (26), giving rise to the appearance of Fibonacci numbers on the surface of phyllotaxis objects.
2. Bodnar's geometry has shown that hyperbolic geometry is much more common in the real world than originally thought. Hyperbolic Fibonacci and Lucas functions, introduced in [41], appear to be fundamental functions of Nature. They arise in various botanic structures, including pine cones, pineapples, cacti, and sunflower heads. Bodnar's new hyperbolic geometry, based on the recursive hyperbolic functions (25) - (28), is of fundamental importance for the future development of the modern life sciences (biology, botany, physiology, medicine, genetics, and so on).
3. There is a fundamental distinction between Lobachevsky's classic hyperbolic geometry and Bodnar's new hyperbolic geometry of phyllotaxis [18-20, 24]. Lobachevsky's geometry is based on the classic hyperbolic functions (22), which use Euler's constant (e) as the base of these functions. The applications of Lobachevsky's geometry relate, first of all, to the "mineral world" and physical phenomena (Einstein's special theory of relativity, four-dimensional Minkowski's world, etc.). Bodnar's geometry, on the other hand, is a hyperbolic geometry of Living Nature. It is based on the recursive hyperbolic Fibonacci and Lucas functions (25) - (28), which use the golden ratio as the base of these functions. In contrast to the classic hyperbolic functions (22), the recursive hyperbolic Fibonacci and Lucas functions (25) - (28) have unique mathematical properties, in particular, recursive properties, similar to Fibonacci and Lucas numbers what give a simple explanation, why Fibonacci and Lucas spirals appear on the surface of phyllotaxis objects.
4. The previous conclusion suggests the importance of Bodnar's geometry for the future development of hyperbolic geometry. As is well established, Lobachevsky's development of hyperbolic geometry was the result of the replacement of Euclid's 5th postulate (the postulate of parallel lines) by a new postulate - "Lobachevsky's postulate." Since then, other new non-Euclidean geometries have emerged by way of "postulate replacement" (e.g. elliptic geometry). In Lobachevsky's time, only one class of hyperbolic functions, determined by the formulas (22), was used widely. The use of these functions pertains to hyperbolic geometry, known as "Lobachevsky's geometry." However, classic hyperbolic functions and classic Lobachevsky's geometry do not

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have recursive properties. Creating a new class of hyperbolic functions, called recursive hyperbolic Fibonacci and Lucas functions [41], became a prerequisite for the derivation of a new kind of hyperbolic geometry called Bodnar's geometry [18-20, 24]. Bodnar's geometry resulted from "hyperbolic function replacement." This suggests a new way of expressing hyperbolic geometry: the search for new hyperbolic functions can lead to new recursive hyperbolic geometries.

5. Furthermore, the recursive hyperbolic Fibonacci functions and Bodnar's geometry are based upon the *golden section*, one of the most beautiful mathematical constants. This provides an important link to Dirac's Principle of Mathematical Beauty.

6. Bodnar's geometry helps to reveal the underlying secret of phyllotaxis - one of the most amazing occurrences in the life sciences. If Nature actually operates according to the model, suggested by Bodnar, then it is reasonable to suggest that Nature acts as a great mathematician by employing recursive hyperbolic Fibonacci and Lucas functions during all existence of Life Nature.

Fibonacci p-numbers and golden p-proportions: the first important generalizations of Fibonacci numbers and Golden Ratio

Pascal's Triangle and Fibonacci p-numbers. In the book [67], the famous American mathematician and populariser of mathematics George Polya (1887 - 1985) has found a surprising connection between Fibonacci numbers and "diagonal sums" of Pascal's triangle (Table 3).

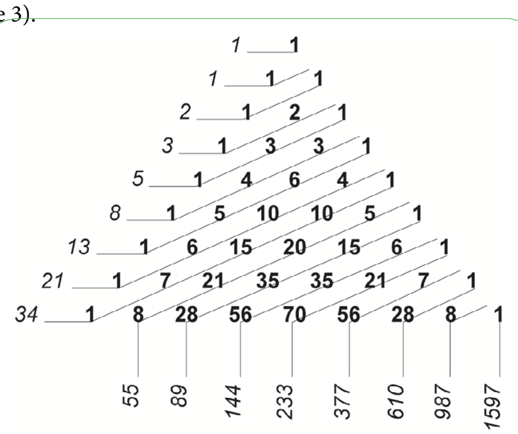

Table 3: Diagonal sums of Pascal's Triangle.

By developing Polya's idea, Alexey Stakhov in the book [33] has obtained wider generalization of Fibonacci numbers. Here we are talking about the new recurrent numerical sequences, introduced in [33] and called the Fibonacci p-numbers. These recurrent numerical sequences for the given are generated by the following general recurrent relation:

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \tag{49}$$

at the seeds:

$$F_p(1) = F_p(2) = \ldots = F_p(p+1) = 1 \tag{50}$$

Note that for the case p=0 the recurrent relation (49) at the seed generates the classic binary numbers and for the case p=1 the classic Fibonacci numbers.

A generalization of the "golden ratio." By studying the Fibonacci p-numbers, given by recurrent relation (49) at the seeds (50), and by considering the limit of the ratio of the neighboring Fibonacci *p*-numbers $\lim\limits_{n\to\infty} \frac{F_p(n)}{F_p(n-1)} = x$, Stakhov derived in [33] the following algebraic equation, which is the characteristic equation for the recurrent relation (49):

$$x^{p+1} - x^p - 1 = 0 \text{ (p=0,1,2,3,...)}. \tag{51}$$

The positive roots of the equation (51) form a set of new mathematical constants Φ_p, which reflect some algebraic properties of Pascal's triangle. The classic golden ratio is a special case of the constants Φ_p for $p = 1$. That is why, the constants Φ_p have been called the *golden p-proportions* [33].

The partial values of Φ_p are showed in Table 4.

p	0	1	2	3	4
Φ_p	2	1.618	1.4656	1.3802	1.3247

Table 4: Partial values of (p=0,1,2,3,4).

This mathematical result led to the admiration of Ukrainian academician Mitropolsky, who wrote the following [68]:

"*Let's ponder upon this result. Within several millennia, starting with Pythagoras and Plato, mankind used the widely known classical Golden Proportion as a unique number. And unexpectedly at the end of the 20th century the Ukrainian scientist Stakhov generalized this result and proved the existence of an infinite number of Golden Proportions! And all of them have the same ability to express Harmony, as well as the classical Golden Proportion. Moreover, Stakhov proved that the golden p-proportions $\Phi_p (1 \le \Phi_p \le 2)$ represented a new class of irrational numbers, which express some otherwise unknown mathematical properties of Pascal triangle. Clearly, such a mathematical result is of fundamental importance for the development of modern science and mathematics.*"

Numeral systems with irrational base: a breakthrough in the theory of numeral systems.

Bergman's system. A story of numeral systems with irrational bases began with a children's computer game that was offered in 1957 by the American 12-year-old prodigy George Bergman. This computer game led him to discovery of very unusual numeral system with the irrational base, called Bergman's system [32]:

$$A = \sum_i a_i \Phi^i, \tag{52}$$

where A is any real number, a_i is a binary numeral {0,1} of the i-th digit, $i = 0, \pm1, \pm2, \pm3, \ldots, \Phi^i$, is the weight of the i-th digit, and $\Phi = (1 + \sqrt{5})/2$) (the golden ratio) is the base of the numeral system (52). On the face of it, there is not any distinction between the formula (52) for Bergman's system and the formula for traditional *binary system*:

$$A = \sum_i a_i 2^i \left(i = 0, \pm1, \pm2, \pm3, \ldots\right) \left(a_i \in \{0,1\}\right), \tag{53}$$

where the digit weights are connected by the following "arithmetic" relations:

$$2^i = 2^{i-1} + 2^{i-1} = 2 \times 2^{i-1}, \tag{54}$$

which underlie "binary arithmetic".

The principal distinction of the numeral system (52) from the binary system (53) is the fact that the irrational number $\Phi = (1 + \sqrt{5})/2)$ (the golden ratio) is used as the base of the numeral system (52) and the digit weights of (52) are connected by the following "arithmetic" relations:

$$\Phi^i = \Phi^{i-1} + \Phi^{i-2} = \Phi \times \Phi^{i-1} \tag{55}$$

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which underlie the "golden arithmetic".

Unfortunately, Bergman's system did not have any consequences for mathematics and computer science because mathematicians and computer experts of that time had recognized Bergman's system as example of "curious mathematical result," which has not exceptional theoretical and applied significance to mathematics and computer science. However, the outstanding American mathematician and world-known expert in computer science Prof. Donald Knuth was the only exception from general mathematical and computer community, because he had referred Bergman's system (52) in his bestseller "The Art of Computer Programming"

Fibonacci p-codes. Fibonacci *p*-codes had been introduced by Alexey Stakhov in his Doctoral thesis "Synthesis of the Optimal Algorithms for Analog-to-Digital Conversion" (1972). The following sum is called *Fibonacci p-code*:

$$N = a_n F_p(n) + a_{n-1} F_p(n-1) + \ldots + a_i F_p(i) + \ldots + a_n F_p(1) \quad (56)$$

where N is natural number, $a_i \in \{0,1\}$ is a binary numeral of the i-th digit of the positional numeral system (56); is a number of bits of the code (56); n is the *Fibonacci p*-number, the weight of the i-th digit of the positional numeral system (56). For the given $p = 0,1,2,3,\ldots$ the *Fibonacci p-numbers* $F_p(i)$ ($i = 1,2,3,\ldots,n$) are given by the recurrent relation (49) at the seeds (50).

Note that the recurrent relation (49) at the seeds (50) generates many remarkable numerical sequences, in particular, the *binary sequence* for the case $p = 0$:

$$1,2,4,8,16,32,64,\ldots,2^{n-1},\ldots, \quad (57)$$

and the classic Fibonacci sequence for the case p=1:

$$1,1,2,3,5,8,13,21,34,55,\ldots, F_n,\ldots \quad (58)$$

The limit of the ratio of two adjacent *Fibonacci p-numbers*

$$\underbrace{\lim}_{n\to\infty} \frac{F_p(i)}{F_p(i-1)} = \Phi_p, \quad (59)$$

is the golden p-proportion, the base of the Fibonacci p-code (56). The limit (59) strives to the mathematical constant Φ_p, the positive root of the algebraic equation (51), which is a generalisation of the "golden" algebraic equation $x^2-x-1 = 0$ with the positive root $\Phi = (1 + \sqrt{5})/2)$ (the golden ratio). That is why, the constants Φ_p ($p = 0,1,2,3,\ldots$) were called the *golden p-proportions.*

Partial cases of the Fibonacci p-codes. Let $P = 0$. For this case, Fibonacci (p=0)-numbers $F_0(i)$ coincide with the "binary" numbers, ie, $F_0(i) = 2^{i-1}$ and therefore the sum (56) takes the form of the classic binary code for natural numbers:

$$N = a_n 2^{n-1} + a_{n-1} 2^{n-2} + \ldots + a_i 2^{i-1} + \ldots + a_1 2^0 \quad (60)$$

Let p=1 . For this case, Fibonacci (p=1)-numbers coincide with the classic Fibonacci numbers, ie, $F_1(i) = F_i$ and for this case the sum (56) takes the following form:

$$N = a_n F_n + a_{n-1} F_{n-1} + \ldots + a_i F_i + \ldots + a_1 F_1 \quad (61)$$

Let now $p = \infty$. For this case all Fibonacci ($p = \infty$)-numbers, given by (49) at the seeds (50) are equal to 1 identically, ie, for any i we have: $F_1(i) = 1$. For this case, the sum (56) takes the following form, called "*unitary code*":

$$N = \underbrace{1+1+\ldots+1}_{N} \quad (62)$$

Thus, the Fibonacci p-codes (56) is a wider generalization of the "binary code" (60) (the case p=0). The classic Fibonacci code (61) (p=1) and "unitary code" (62) (p=∞) are partial cases of the Fibonacci p-codes (56).

The *Fibonacci p-codes* and following from them *Fibonacci arithmetic* have been described in the first Stakhov's book [33]. This book is pioneering book in this field and opens a new stage in computer science, *Fibonacci computers*, based on the *Fibonacci p-codes* (56).

All technical developments in the field of "Fibonacci Computers" were protected by 65 patents of the USA, Japan, England, France, Germany, Canada and other countries what confirms Stakhov's priority in "Fibonacci Computers" as new direction in specialized computers for mission-critical applications.

Codes of the Golden p-proportions. Let us consider the binary code for real numbers:

$$A = \sum_i a_i 2^i \left(i = 0, \pm1, \pm2, \pm3, \ldots\right) \quad (63)$$

where the digit weights $2^i \left(i = 0, \pm1, \pm2, \pm3, \ldots\right)$ are connected with the following well-known identities:

$$2^i = 2^{i-1} + 2^{i-1} \left(summing\ identity\right) \quad (64)$$

$$2^i = 2 \times 2^{i-1} \left(multiplicative\ identity\right) \quad (65)$$

The binary code (63) allows the following generalization. Consider the set of the following standard line segments:

$$\{\ldots, \Phi_p^n, \Phi_p^{n-1}, \ldots, \Phi_p^{n-p-1}, \ldots, \Phi_p^0 = 1, \Phi_p^{-1}, \ldots, \Phi_p^{-k}, \ldots\} \quad (66)$$

where $\Phi_p \left(p = 0,1,2,3,\ldots\right)$ are the golden p-proportions, which are the real roots of the golden p-ratio equation (51). The powers of the golden p-proportions $\Phi_p^n \left(p = 0,1,2,3,\ldots; n = 0, \pm1, \pm2, \pm3, \ldots\right)$ are connected by the remarkable identities:

$$\Phi_p^n = \Phi_p^{n-1} + \Phi_p^{n-p-1} \left(summing\ identity\right) \quad (67)$$

$$\Phi_p^n = \Phi_p \times \Phi_p^{n-1} \left(multiplicative\ identity\right) \quad (68)$$

By using (66), we can get the following positional binary (0,1) method of real numbers representation called the *code of the golden p-proportions*:

$$A = \sum_i a_i \Phi_p^i;\ p = 0,1,2,3,\ldots;\ i = 0, \pm1, \pm2, \pm3, \ldots \quad (69)$$

where A is real number, $a_i \in \{0,1\}$ is the bit of the i-th digit; Φ_p^i is the weight of the i-th digit; the golden p-proportion is the base of the numeral system (69).

As it is known, the binary system (63) is considered as definition of real numbers. Notice that the *codes of the golden p-proportions* (69) are a generalization of the binary code (63) and consequently they may be regarded as unusual definitions of real numbers. For the cases p>0, the bases of the the *codes of the golden p-proportions* (69) are irrational numbers Φ_p^i . Such approach to the *codes of the golden p-proportions* (69) puts forward the mathematical constants Φ_p^i on the first place in number theory what could lead to new theoretical-numerical results in number theory.

Theory of the codes of the golden p-proportions (69) and their applications in *number theory, computer science* and *digital metrology* has been described in the books [2, 34].

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4.5. ***Partial cases of the codes of the golden p-proportions***. Let us consider the partial cases of the codes of the golden *p*-proportions (69). For the case *p*=0, the formula (69) is reduced to the formula (63) for the binary system and for the case p=1 to Bergman's system (52).

Note that all mathematical constants Φ_p^i (with the exception of the case *p*=0) are irrational numbers. This means that the codes of the golden p-proportions (69) are expanding a class of numeral systems with irrational bases indefinitely. This result is great interest for number theory because the codes of the golden *p*-proportions (69) can be considered as new definition of real numbers. On the other hand, the codes (69) and the resulting from them "golden arithmetic" are of the interest to computer science and digital metrology.

The Fibonacci *p*-codes and codes of the golden *p*-proportions are described in the article [39], where the computer aruithmetic for these codes have been described and discussed the main advantages of usage of these codes in computer science for detection of errors in digital structures.

Ternary mirror-symmetrical arithmetic. In 2002 "The Computer Journal" (British Computer Society) had published Stakhov's article [35]. The main purpose of this article was to develop *ternary mirror-symmetrical arithmetic based on Bergman' system and ternary principle*, used by Russian engineer Nikolay Brousentsov (1925 —2014) for designing the first in computer history ternary computer "Setun" (Moscow University). Stakhov's article [35] has great theoretical and applied interest both for computer science and digital metrology. It opens a new way for designing of noise-immune ternary processors and specialized ternary computers for mission-critical applications.

It is no coincidence that this article [35] caused a positive reaction from the Western computer community. The prominent American mathematician and a world-renowned expert in computer science Donald Knuth was the first outstanding scientist who congratulated the author with the publication of this article. In his letter, he informed the author about his intention to include a description of the *ternary mirror-symmetrical arithmetic* into the new edition of his bestseller "*Art of Computer Programming.*"

Note that the descryption of the ternary mirror-symmetrical arithmetic and its connection to Bergman's system is given in Stakhov's 2015 article [40].

The "golden" number theory and new properties of natural numbers. In 2015 the "British Journal of Mathematics and Computer Science" has published Stakhov's article [40], which is continuation and development of the article [35].

Let us consider the representation of natural numbers *N* in Bergman's system, Φ-called of natural number *N*:

$$\Phi - code : N = \sum_i a_i \Phi^i \left(i = 0, \pm 1, \pm 2, \pm 3, \ldots \right) \quad (70)$$

A study of (70) leads us to the following far not trivial mathematical results, which can be formulated as the following theorem [38].

Theorem 1. All natural numbers can be represented in Bergman's system (52) by using a finite number of binary numerals.

This result can be generalized for the codes of the golden p-proportions (69) as follows.

Theorem 2. For the given p>0, all natural numbers can be represented in the golden p-proportion code (70) by using the finite number of bits.

Bergman's system (52) and codes of the golden p-proportions (69) are sources for new number-theoretical results. The Z- and D-properties are the most surprising among them.

In Fibonacci numbers theory [4], there is the following remarkable formula:

$$\Phi^i = \frac{L_i + F_i\sqrt{5}}{2} \left(i = 0, \pm 1, \pm 2, \pm 3, \ldots \right)$$

which express the golden ratio powers Φ^i through "extended" Lucas number L_i and Fibonacci numbers F_i.

By using (71), we can prove the following theorems [38].

Theorem 3 (Z-property). If we represent an arbitrary natural number *N* in the Φ-code (70) and then substitute the "extended" Fibonacci numbers F_i (20) instead of the golden ratio power Φ^i in the formula (70), where the index *i* takes its values from the set {0,±1,±2,±3…}, then the sum that appear as a result of such a substitution is equal to 0 **identically**, independently on the initial natural number *N*, that is,

$$For\ any\ N = \sum_i a_i \Phi^i\ after\ substitution\ F_i \to \Phi^i :$$
$$\sum_i a_i F_i \equiv 0 \left(i = 0, \pm 1, \pm 2, \pm 3, \ldots \right) \quad (72)$$

Theorem 4 (*D*-property). If we represent an arbitrary natural number N in the -code (70) and then substitute the "extended" Lucas numbers Li (21) instead of the golden ratio power   in the formula (70), where the index *i* takes its values from the set {0,±1,±2,±3…}, then the sum that appears as a result of such a substitution is equal to *2N* **identically**, independently of the initial natural number *N*, that is,

$$For\ any\ N = \sum_i a_i \Phi^i\ after\ substitution\ F_i \to \Phi^i :$$
$$\sum_i a_i L_i \equiv 2N \left(i = 0, \pm 1, \pm 2, \pm 3, \ldots \right) \quad (73)$$

In conclusion, we note that Theorems 1–4, formulated above, are valid only for natural numbers. Therefore, we have a right to consider the results of Theorems 1–4 as new properties of natural numbers. This means that we found new, previously unknown properties of natural numbers, the theoretical study of which began 2.5 millennia ago, at least starting from Euclid's *Elements*. These properties are of great interest for number theory and can be used in computer science.

Fibonacci λ-numbers and "metallic means": the second important generalizations of Fibonacci numbers and Golden Ratio

Fibonacci and Lucas λ-numbers: Let us give a real number λ >0 and consider the following recurrent relation:

$$F_\lambda\left(n+2\right) = \lambda F_\lambda\left(n+1\right) + F_\lambda\left(n\right)$$

for the initial terms (seeds):

$$F_\lambda\left(0\right) = 0,\ F_\lambda\left(1\right) = 1.$$

The recurrent relation (74) at the seeds (75) generates an infinite number of new recurrent numerical sequences, because every real number   generates its own numerical sequence.

Let us consider the partial cases of the recurrent relation (74). For the case   the recurrent relation (74) and the seeds (75) are reduced to the following:

$$F_1(n+2)=F_1(n+1)+F_1(n) \quad (76)$$

$$F_1(0)=0,\ F_1(1)=1 \quad (77)$$

The recurrent relation (76) at the seeds (77) generates the classical Fibonacci numbers:

$$0,1,1,2,3,5,8,13,21,34,... \quad (78)$$

Based on this fact, we will name a general class of the recurrent numerical sequences, generated by the recurrent relation (74) at the seeds (75), the *Fibonacci λ-numbers.*

For the case λ=2 the recurrent relation (74) at the seeds (75) are reduced to the following:

$$F_2(n+2)=2F_2(n+1)+F_2(n) \quad (79)$$

$$F_2(0)=0,\ F_2(1)=1 \quad (80)$$

The recurrent relation (79) at the seeds (80) generates the so-called *Pell numbers* [69]:

$$0,1,2,5,12,29,70,169,408,... \quad (81)$$

The Fibonacci λ-numbers have many remarkable properties, similar to the properties of the classic Fibonacci numbers. It easy to prove that the Fibonacci λ-numbers, as well as the classic Fibonacci numbers, can be "extended" to negative values of the discrete variable *n*.

Table 5 shows the four extended Fibonacci λ-sequences, corresponding to the cases of λ=1,2,3,4.

n	0	1	2	3	4	5	6	7	8
$F_1(n)$	0	1	1	2	3	5	8	13	21
$F_1(-n)$	0	1	–1	2	–3	5	–8	13	–21
$F_2(n)$	0	1	2	5	12	29	70	169	408
$F_2(-n)$	0	1	–2	5	–12	29	–70	169	–408
$F_3(n)$	0	1	3	10	33	109	360	1189	3927
$F_3(-n)$	0	1	–3	10	–33	109	–360	1199	–3927
$F_4(n)$	0	1	4	17	72	305	1292	5473	23184
$F_4(-n)$	0	1	–4	17	–72	305	–1292	5473	–23184

Table 5: The extended Fibonacci λ-numbers (λ=1,2,3,4)

In the article [44] the following recurrent relation and the seeds for the Lucas λ-numbers have been obtained:

$$L_\lambda(n)=\lambda L_\lambda(n-1)+L_\lambda(n-2) \quad (82)$$

$$L_\lambda(0)=2,\ L_\lambda(1)=\lambda \quad (83)$$

The generalized Cassini formula. For the Fibonacci λ-numbers (the cases λ =1.2.3…), the well-known Cassini's formula (18) can be generalized as follows:

$$F_\lambda^2(n)-F_\lambda(n-1)F_\lambda(n+1)=(-1)^{n+1} \quad (84)$$

This formula sounds as follows:

"The quadrate of any Fibonacci λ-number for the given integer λ= 1.2.3... *is always different from the product of the two adjacent Fibonacci λ-numbers* $F_\lambda(n-1)$ and $F_\lambda(n+1)$, *which differ from the initial Fibonacci λ-number* $F_\lambda(n)$ *by 1; herein the sign of the difference of 1 depends on the parity of n: if n is even, then the difference of 1 is taken with the "minus" sign, otherwise, when n is odd, then with the "plus" sign."*

Until now, we have assumed that only the classic Fibonacci numbers have the unusual property, determined by Cassini's formula (18). It is proved, that there are an infinite number of such numerical sequences. All the Fibonacci λ-numbers, generated by the recurrent relation (74) at the seeds (75) for the given integers λ= 1.2.3... have the same property, determined by the generalized Cassini formula (84)!

It is well known that the study of integer sequences lies within the area of number theory. The Fibonacci λ-numbers are integers for the cases. Therefore, for many mathematicians in the field of number theory, the existence of an infinite number of integer sequences, which satisfy the unique property, determined by the generalized Cassini formula (84), may be a great surprise!

***The "metallic means" by Vera de Spinadel*:** The Argentinian mathematician Vera de Spinadel [6] and French mathematician of the Egyptian origin Modhat Gazale [7] have intrtoduced the most contribution into the development of the Fibonacci λ-numbers theory and their applications.

Let us consider the cases λ= 1.2.3... and represent the recurrent relation (74) as follows:

$$\frac{F_\lambda(n+2)}{F_\lambda(n+1)}=\lambda+\frac{1}{\frac{F_\lambda(n+1)}{F_\lambda(n)}} \quad (85)$$

For the case $n\to\infty$ the expression (85) is reduced to the following quadratic equation:

$$x^2-\lambda x-1=0, \quad (86)$$

which is the characteristic equation for the Fibonacci λ-numbers.

The equation (86) has two roots, the positive root:

$$x_1=\frac{\lambda+\sqrt{4+\lambda^2}}{2} \quad (87)$$

and the negative root

$$x_2=\frac{\lambda-\sqrt{4+\lambda^2}}{2} \quad (88)$$

Denote the positive root x_1 by, Φ_λ that is,

$$\Phi_\lambda=\frac{\lambda+\sqrt{4+\lambda^2}}{2} \quad (89)$$

Note that for the case λ=1 the formula (89) is reduced to the formula for the golden ratio:

$$\Phi_1=\frac{1+\sqrt{5}}{2} \quad (90)$$

This means that the formula (89) set forth a wide class of new mathematical constants (89), which is a generalization of the golden ratio (90).

Based on this analogy, Vera de Spinadel in the book [6] named the group of the mathematical constants (89) for the cases the metallic means, also known as the metallic proportions. If we take in (89), we then get the following mathematical constants, which, according to de Spinadel, have the special names:

$$\Phi_1=\frac{1+\sqrt{5}}{2}\ (\text{the Golden Mean}, \lambda=1);$$

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$$\Phi_2 = 1+\sqrt{2}\ (\text{the Silver Mean},\ \lambda = 2);$$
$$\Phi_3 = \frac{3+\sqrt{13}}{2}\ (\text{the Bronze Mean},\ \lambda = 3);$$
$$\Phi_4 = 2+\sqrt{5}\ (\text{the Copper Mean},\ \lambda = 4).$$

Other metallic means (λ≥5) have not special names:

$$\Phi_5 = \frac{5+\sqrt{29}}{2};\quad \Phi_6 = 3+2\sqrt{10};\quad \Phi_7 = \frac{7+2\sqrt{14}}{2};\quad \Phi_8 = 4+\sqrt{17}. \tag{91}$$

The simplest algebraic properties of the metallic means. Let us represent the root x_2, given by (88), through the metallic mean (89). After simple transformation we can write the root x_2 as follows:

$$x_2 = \frac{\lambda-\sqrt{4+\lambda^2}}{2} = \frac{\left(\lambda-\sqrt{4+\lambda^2}\right)\left(\lambda+\sqrt{4+\lambda^2}\right)}{2\left(\lambda+\sqrt{4+\lambda^2}\right)} == \frac{-4}{2\left(\lambda+\sqrt{4+\lambda^2}\right)} = -\frac{1}{\Phi_\lambda} \tag{92}$$

By using (86), (89) and (92), it is easy to prove the following identity:

$$\Phi_\lambda^n = \lambda\Phi_\lambda^{n-1} + \Phi_\lambda^{n-2} \tag{93}$$

where n=0,±1, ±2, ±3….

By using the identities $\Phi_\lambda^2 = 1+\lambda\Phi_\lambda$, $\Phi_\lambda = \sqrt{1+\lambda\Phi_\lambda}$, and $\Phi_\lambda = \lambda + \frac{1}{\Phi_\lambda}$ following from (93), we can get the following surprising representations of the metallic means Φ_λ in the form of *nested radical* and *continued fraction:*

$$\Phi_\lambda = \sqrt{1+\lambda\sqrt{1+\lambda\sqrt{1+\lambda\sqrt{1+\ldots}}}} \tag{94}$$

$$\Phi_\lambda = \lambda + \cfrac{1}{\lambda + \cfrac{1}{\lambda + \cfrac{1}{\lambda+\ldots}}} \tag{95}$$

Note that for the case λ=1 the representations (94) and (95) are similar by their mathematical structure to the following well-known *nested radical* and *continued fraction* representations for the classic golden ratio:

$$\Phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}};\quad \Phi = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}} \tag{96}$$

The representations of the "metallic means" in the forms of the *nested radical* (94) and *continued fraction* (95), similar to the golden ratio's surprising representations (96), are the additional confirmation of the fact that the *metallic means* Φ_λ are new and striking constants of mathematics!

Gazale's formulas. The formulas (74), (75) and (82), (83) define the Fibonacci and Lucas λ-numbers $F_\lambda(n)$ and $L_\lambda(n)$ recursively. It is derived in [44] the following formulas for the analytic representation of the "extended" Fibonacci and Lucas λ-numbers $F_\lambda(n)$ and $L_\lambda(n)$ in the explicit form through the metallic means Φ_λ(λ=1,2,3):

$$F_\lambda(n) = \begin{cases} \dfrac{\Phi_\lambda^n - \Phi_\lambda^{-n}}{\sqrt{4+\lambda^2}} & \text{for } n = 2k \\ \dfrac{\Phi_\lambda^n + \Phi_\lambda^{-n}}{\sqrt{4+\lambda^2}} & \text{for } n = 2k+1 \end{cases} \tag{97}$$

$$L_\lambda(n) = \begin{cases} \Phi_\lambda^n - \Phi_\lambda^{-n} & \text{for } n = 2k+1 \\ \Phi_\lambda^n + \Phi_\lambda^{-n} & \text{for } n = 2k \end{cases} \tag{98}$$

These formulas were named in [44] *Gazale's formulas* after Midchat Gazale who first introduced analytic representation of the "extended" Fibonacci λ-numbers $F_\lambda(n)$ [7].

Note that for the case λ=1 the Gazale's formulas (97), (98) are reduced to Binet's formulas (20), (21).

Gazale's hypothesis. The central idea of Gazale's book [7] is the notion of *self-similarity*. Gazale was one of the first who begun to study Fibonacci λ-numbers. The Gazale's formula, which expresses Fibonacci λ-numbers through the "metallic means," is one of the main Gazale's mathematical achievements, described in the book [7].

In the book [7], Gazale put forward the following unusual hypothesis, which has direct relation to mathematical models of *self-similarity*:

Gazale's hypothesis: "*The numerical sequence* $F_{m,n+2} = F_{m,n} + mF_{m,n+1}$ *which I call here the Fibonacci sequence of the order m, play a key role in the study of self-similarity.*"

If we take in this formula that $m = \lambda$, $F_{m,n+2} = F_\lambda(n+2)$, $F_{m,n} = F_\lambda(n)$, then we get the recurrent relation (74) for the Fibonacci λ-numbers.

This means that the recurrent relation (74), which sets forth the Fibonacci λ-numbers, according to *Gazale's hypothesis*, expresses the *self-similarity principle*, which is one of the most important principles of Nature, scienc and mathematics.

Hyperbolic Fibonacci and Lucas λ-functions

Definition. Gazale's formulas (97) and (98) are the source for the introduction of a new class of hyperbolic functions.

Hyperbolic Fibonacci λ-sine and λ-cosine

$$sF_\lambda(x) = \frac{\Phi_\lambda^x - \Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}} == \frac{1}{\sqrt{4+\lambda^2}}\left[\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x - \left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}\right] \tag{99}$$

$$cF_\lambda(x) = \frac{\Phi_\lambda^x + \Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}} == \frac{1}{\sqrt{4+\lambda^2}}\left[\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x + \left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}\right] \tag{100}$$

Hyperbolic Lucas λ-sine and λ-cosine

$$sL_\lambda(x) = \Phi_\lambda^x - \Phi_\lambda^{-x} = \left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x - \left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x} \tag{101}$$

$$cL_\lambda(x) = \Phi_\lambda^x + \Phi_\lambda^{-x} = \left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x + \left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x} \tag{102}$$

where x is a continuous variable and λ is a given integer λ = 1,2,3, ….

It is easy to see that the functions (99), (100) and (101), (102) are connected by the simple relations:

$$sF_\lambda(x) = \frac{sL_\lambda(x)}{\sqrt{4+\lambda^2}};\quad cF_\lambda(x) = \frac{cL_\lambda(x)}{\sqrt{4+\lambda^2}}. \tag{103}$$

Comparing the formula (97) with the formulas (99), (100) and then the formula (98) with the formulas (101), (102), it is easy to prove the following fundamental relationships of hyperbolic Fibonacci and Lucas λ-functions with Fibonacci and Lucas λ-numbers, given by the Gazale formulas (97), (98):

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$$F_\lambda(n)=\begin{cases} sF_\lambda(n) \text{ for } n=2k \\ cF_\lambda(n) \text{ for } n=2k+1 \end{cases} \quad (104)$$

$$L_\lambda(n)=\begin{cases} cL_\lambda(n) \text{ for } n=2k \\ sL_\lambda(n) \text{ for } n=2k+1 \end{cases}, \quad (105)$$

where k takes the values from the set $k=0,\pm1,\pm2,\pm3,\ldots$.

The partial cases of the hyperbolic Fibonacci and Lucas λ-functions: the case λ=1 the golden ratio (90) is the base of the hyperbolic Fibonacci and Lucas λ-functions (99)-(102), which are reduced to the above recursive hyperbolic Fibonacci and Lucas functions (25)–(28). We will name the functions (25)–(28) the "*golden*" *hyperbolic Fibonacci and Lucas functions*.

For the case λ=2 , the silver mean ($\Phi_2=1+\sqrt{2}$) is the base of a new class of hyperbolic functions. We will name them the "silver" hyperbolic Fibonacci and Lucas functions:

$$sF_2(x)=\frac{\Phi_2^x-\Phi_2^{-x}}{\sqrt{8}}=\frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^x-\left(1+\sqrt{2}\right)^{-x}\right] \quad (106)$$

$$cF_2(x)=\frac{\Phi_2^x+\Phi_2^{-x}}{\sqrt{8}}=\frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^x+\left(1+\sqrt{2}\right)^{-x}\right] \quad (107)$$

$$sL_2(x)=\Phi_2^x-\Phi_2^{-x}=\left(1+\sqrt{2}\right)^x-\left(1+\sqrt{2}\right)^{-x} \quad (108)$$

$$cL_2(x)=\Phi_2^x+\Phi_2^{-x}=\left(1+\sqrt{2}\right)^x+\left(1+\sqrt{2}\right)^{-x} \quad (109)$$

In the article [44], the mathematical formulas for the "bronze" (λ=3) and the "copper" (λ=4) hyperbolic Fibonacci and Lucas λ-functions have been derived.

Note that the list of these functions can be continued ad infinitum.

Hyperbolic and recursive properties of the hyperbolic Fibonacci and Lucas λ-functions:

As examples of the hyperbolic properties of the functions (99)–(102), we consider the parity properties and the analog of the Pythagoras Theorem:

Parity properties

$$\begin{aligned} &sF_\lambda(-x)=-sF_\lambda(x);\ cF_\lambda(-x)=cF_\lambda(x) \\ &sL_\lambda(-x)=-sL_\lambda(x);\ cL_\lambda(-x)=cL_\lambda(x) \end{aligned} \quad (110)$$

Analogs of the Pythagoras Theorem

$$\begin{aligned} &\left[cF_\lambda(x)\right]^2-\left[sF_\lambda(x)\right]^2=\frac{4}{4+\lambda^2} \\ &\left[cL_\lambda(x)\right]^2-\left[sL_\lambda(x)\right]^2=4 \end{aligned} \quad (111)$$

Some *recursive properties* of the functions (99)–(102) are given by the following theorems, proved in [2, 44].

Theorem 5: The following relations, which are similar to the recurrent relation for the Fibonacci λ-numbers $F_\lambda(n+2)=\lambda F_\lambda(n+1)+F_\lambda(n)$, are valid for the hyperbolic Fibonacci λ-functions:

$$\begin{aligned} &sF_\lambda(x+2)=\lambda cF_\lambda(x+1)+sF_\lambda(x), \\ &cF_\lambda(x+2)=\lambda sF_\lambda(x+1)+cF_\lambda(x). \end{aligned} \quad (112)$$

Theorem 6: (the generalized Cassini's formula for continuous domain) following relations, which are similar to the generalized Cassini's formula for the Fibonacci λ-numbers $F_\lambda^2(n)-F_\lambda(n-1)F_\lambda(n+1)=(-1)^{n+1}$, are valid for the hyperbolic Fibonacci λ-functions:

$$\begin{aligned} &\left[sF_\lambda(x)\right]^2-cF_\lambda(x+1)cF_\lambda(x-1)=-1, \\ &\left[cF_\lambda(x)\right]^2-sF_\lambda(x+1)sF_\lambda(x-1)=1. \end{aligned} \quad (113)$$

Unique properties of the hyperbolic Fibonacci and Lucas λ-functions: It should be noted the following unique properties of the hyperbolic Fibonacci and Lucas λ- functions (99)–(102):

1. The hyperbolic Fibonacci and Lucas λ-functions (99)–(102) are, on the one hand, a generalization of the classic hyperbolic functions (22), but on the other hand, a generalization of the recursive hyperbolic Fibonacci and Lucas functions (25)–(28), which are a partial cases of the functions (99)–(102) for λ=1.

2. Their uniqueness consists of the fact that they, on the one hand, retain all *hyperbolic properties*, inherent for the classic hyperbolic functions (22). On the other hand, they have *recursive properties*, inherent to the recursive hyperbolic Fibonacci and Lucas functions (25)–(28).

3. The next unique feature of the functions (99)–(102) is the fact that the general formulas (99)–(102) define theoretically infinite number of new classes of the recursive hyperbolic functions, because every integer generates a new, previously unknown class of the recursive hyperbolic functions. Really, we are talking about general theory of the recursive hyperbolic functions.

4. One more unique feature of the functions (99)–(102) is their deep connection to the "extended" Fibonacci and Lucas λ-numbers, defined by *Gazale's formulas* (97), (98). This connection is determined identically by the relations (104), (105).

5. According to *Gazale's hypothesis*, the recursive hyperbolic λ-functions (99)–(102), following from *Gazale's formulas* (97), (98), express the similarity principle, which is the most important principle of Nature, science and mathematics.

An original solution of Hilbert's Fourth Problem as way to the "Golden" Hyperbolic Geometry

A general idea: Above we have represented wide generalization of the recursive hyperbolic Fibonacci and Lucas functions (25)–(28). Here the recursive hyperbolic Fibonacci and Lucas λ-functions (99)–(102), which extend the class of the *recursive hyperbolic functions* ad infinitum, are described. These new classes of the surprising recursive hyperbolic functions, based on Spinadel's *metallic means* (89) and *Gazale's formulas* (97), (98) are used as the basis of the original solution to Hilbert's Fourth Problem.

It follows from this statement that the number of new hyperbolic geometries, following from such approach, is theoretically infinite. We will name these new recursive hyperbolic geometries, based on the self-similarity principle, by the common title of the "Golden" Hyperbolic Geometry. A theory of the "Golden" Hyperbolic Geometry are stated in the book [26].

Thus, the "Golden" Hyperbolic Geometry has two distinctive features:

1. This geometry is recursive geometry, based on the recursive Fibonacci and Lucas λ-functions (99) – (102).

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2. The principle of self-similarity underlies this new hyperbolic geometry.

The metric λ-forms of Lobachevski's plane: As is known, the classic metric form of Lobachevski's plane with pseudo-spherical coordinates (u,v), $0 < u < +\infty$, $0 < v < +\infty$ which has a Gaussian curvature $k=-1$, is given with the following formula [26]:

$$(ds)^2 = (du)^2 + sh^2(u)(dv)^2 \qquad (114)$$

where ds is an element of length, $sh(u)$ is a classic hyperbolic sine from (22). It is clear that the classical hyperbolic sine plays a key role in the metric form of Lobachevski's plane (114).

$$(ds)^2 = \ln^2(\Phi_\lambda)(du)^2 + \frac{4+\lambda^2}{4}\left[sF_\lambda(u)\right]^2(dv)^2 \qquad (115)$$

where $\Phi_\lambda = \frac{\lambda + \sqrt{4+\lambda^2}}{2}$ is the "metallic proportion" (89) and $sF_\lambda(u)$ is the hyperbolic Fibonacci λ-sine (99). The forms (115) are called the metric -forms of Lobachevski's plane [26].

The formula (115) gives an infinite number of new *Lobachevski's* geometries ("*golden,*" "*silver,*" "*bronze,*" "*cooper*" and so on ad infinitum) according to the used class of the recursive hyperbolic Fibonacci λ-functions (99), (100). This means that there is infinite number of the new hyperbolic geometries, which are based on the *metallic means (89).* These new hyperbolic geometries "*with equal right*, *stand next to Euclidean geometry*" (David Hilbert). Thus, the formula (115) can be considered as the original solution to Hilbert's Fourth Problem. There are an infinite number of the new hyperbolic geometries, described by the formula (115), which are close to Euclidean geometry. Every of these geometries manifests itself in Fibonacci λ-numbers (74), which can appear in physical world similarly to *Bodnar's hyperbolic geometry* [18-20, 24], which explains why "Fibonacci spirals" appear at the surface of phyllotaxis' objects.

Detailed comparative analyzes of the formulas (114) and (115) is given in the book [26].

An original solution of Hilbert's Fourth Problem as way to the "Golden" Hyperbolic Geometry

A new challenge to theoretical natural sciences. Thus, the main result of the researches, described in [26], is the proof of the existence of an infinite number of the recursive hyperbolic λ-functions (99) - (102), based on the metallic means (89). Note, that every type of the recursive hyperbolic λ-functions, determined by (99) - (102), generates for the given λ=1,2,3,.. its own kind of recursive hyperbolic geometry what leads to the appearance of the "physical worlds" with specific properties, which are determined by the metallic means (89). The new geometric theory of phyllotaxis, created by Oleg Bodnar [18-20, 24], is the striking example of this. Bodnar proved that the "*world of phyllotaxis*" is a specific "*hyperbolic world,*" in which the hyperbolicity manifests itself in the Fibonacci spirals on the surface of phyllotaxis' objects.

However, the "golden" hyperbolic Fibonacci functions (25), (26), which underlie the "*hyperbolic world of phyllotaxis*" [18-20, 24], are a special case of the recursive hyperbolic Fibonacci λ-functions (99), (100) for λ=1. In this regard, we have all reasons to suppose that other types of the recursive hyperbolic λ-functions (99), (100), based on the *metallic means,*" can be good models for the new "*hyperbolic worlds*" that can really exist in Nature. Modern science cannot find these special "hyperbolic worlds," because the recursive hyperbolic functions (99), (100) were unknown until now [41,44,45]. Based on the success of *Bodnar's hyperbolic geometry* [18-20, 24], we can put forward in front to theoretical physics, chemistry, crystallography, botany, biology, genetics and other branches of theoretical natural sciences the *challenge for searching of the new hyperbolic worlds of Nature, based on the new types of the recursive hyperbolic λ-functions (99), (100).*

However, the process of finding new hyperbolic worlds of Nature, based on recursive hyperbolic Fibonacci and Lucas λ-functions, carried out intensively in modern theoretical physics. This is confirmed by the latest research in this area, as described in the articles [28-31].

Studying the recursive hyperbolic functions (99), (100), we can assume [26] that the recursive hyperbolic λ-functions with the bases

$$\Phi_1 = \frac{1+\sqrt{5}}{2}\ (\text{the Golden Mean, } \lambda = 1);$$

$$\Phi_2 = 1+\sqrt{2}\ (\text{the Silver Mean , } \lambda = 2);$$

$$\Phi_3 = \frac{3+\sqrt{13}}{2}\ (\text{the Bronze Mean, } \lambda = 3);$$

$$\Phi_4 = 2+\sqrt{5}\ (\text{the Cooper Mean, } \lambda = 4).$$

are of the greatest interest for theoretical physics and in general for theoretical natural sciences.

The "*golden*" *recursive hyperbolic Fibonacci functions*

$$sF(x) = \frac{\Phi^x - \Phi^{-x}}{\sqrt{5}};\quad cF(x) = \frac{\Phi^x + \Phi^{-x}}{\sqrt{5}}, \qquad (116)$$

based on the classic *golden ratio*, plays the leading role among them. These functions underlie *Bodnar's geometry* [18-20, 24], golden differential geometry [28], and *golden Riemannian manifolds* [29, 30].

The next candidate for the new "hyperbolic world" of Nature (after "Bodnar's hyperbolic geometry" [18-20, 24]) may be, for example, *silver hyperbolic functions*:

$$sF_2(x) = \frac{\Phi_2^x - \Phi_2^{-x}}{\sqrt{8}} = \frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^x - \left(1+\sqrt{2}\right)^{-x}\right];$$
$$cF_2(x) = \frac{\Phi_2^x + \Phi_2^{-x}}{\sqrt{8}} = \frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^x + \left(1+\sqrt{2}\right)^{-x}\right], \qquad (114)$$

which are connected with *Pell numbers* [69] and are based on the silver proportion $\Phi_2=1+\sqrt{2}$, connected with the fundamental mathematical constant $\sqrt{2}$ [26].

The formula (115) gives an infinite number of new *Lobachevski's geometries* ("*golden,*" "*silver,*" "*bronze,*" "*cooper*" and so on ad infinitum) according to the used class of the recursive hyperbolic Fibonacci λ-functions (99), (100). This means that there is infinite number of the new hyperbolic geometries, which are based on the *metallic means* (89). These new hyperbolic geometries "*with equal right, stand next to Euclidean geometry*" (David Hilbert). Thus, the formula (115) can be considered as the original solution to Hilbert's Fourth Problem. There are an infinite number of the new hyperbolic geometries, described by the formula (115), which are close to Euclidean geometry. Every of these geometries manifests itself in Fibonacci λ-numbers (74), which can appear in physical world similarly to *Bodnar's hyperbolic geometry* [18-20, 24], which explains why "Fibonacci spirals" appear at the surface of phyllotaxis' objects.

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which are connected with *Pell numbers* [69] and are based on the silver proportion $\Phi_2=1+\sqrt{2}$, connected with the fundamental mathematical constant $\sqrt{2}$[26].

This assumption is confirmed by Mustafa Özkan and Betül Peltek's 2016 article [27], published in the International Electronic Journal of Geometry. The main novelty of the paper [27] is to study the geometry of the silver structure based on silver proportion $\Phi_2=1+\sqrt{2}$. It should be noted that silver hyperbolic functions are also examined in Bodnar's article [70].

The article [31] describes the *metallic Riemannian manifolds*, based on the use of Spinadel *metallic proportions*. The article [31] is closest by its ideas to author's book [26] and article [49].

The main conclusion of this reasoning consists in the fact that in modern scientific community there has been formed a group of scientists from different countries (USA, Canada, Ukraine, Russia, England, France, Argentina, Turkey, Romania and other countries), which in their works [1-59] support the "golden" paradigm of the ancient Greeks about Universe Harmony, based on the "golden ratio" and "metallic proportions" of Vera W. de Spinadel.

The most striking examples of contemporary scientific discoveries based on the "golden" paradigm of the ancient Greeks

Besides Bodnar's geometry [18-20, 24], we can give other striking examples of applications of the "golden" paradigm of the ancient Greeks in modern theoretical natural science.

"***Parquet's problem" and Penrose's tiles***. The English mathematician Sir Roger Penrose was the first researcher, who found an original solution of the "*parquet's problem*" known from ancient times. In 1972, he has covered a planar surface in non-periodic manner, by using only two simple polygons. In the simplest form, *Penrose's tiling* [53] is a non-random set of rhombi of two types, which follow directly from the regular pentagon and pentagram, based on the *golden ratio.* The first one, called "thick" rhombus (Figure 2a), has the internal angles 72° and 108° and the second one (Figure 2b), called "thin" rhombus, has the internal angles 36° and 144°.

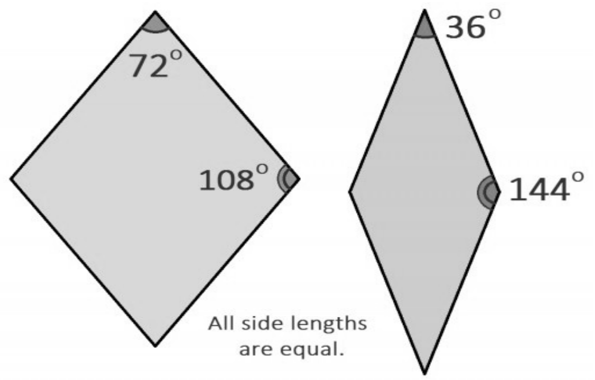

Figure 2: Penrose's rhombi: "thick" rhombus (a) and "thin" rhombus (b) . (Wikipedia, the free encyclopedia, from . https://www.google.ca/?gws_rd=ssl#q=penrose+rhombus)

As Sir Roger Penrose proved, the "thin" and "thick" rhombi in Figure 2 allow covering completely an infinite planar surface. Below in Figure 3, we can see a process of sequential constructing of Penrose's tiling by using the "thick" and "thin" rhombi (Figure 2).

It is proved that the ratio of the number of the "thick" rhombi (Figure 2a) to the number of the "thin" rhombi (Figure 2b) in Penrose's tiling (Figure 3) strives to the golden ratio in the limit that means that the Penrose's tiles are based on the "golden" paradigm of the ancient

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Greeks. Note that the Penrose tiling [53] is of extremely important for crystallography.

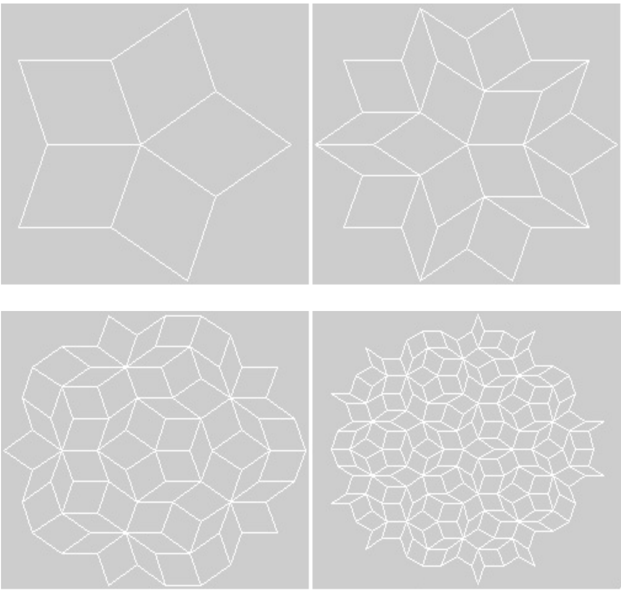

Figure 3: Penrose's tiling.
(Wikipedia, the free encyclopedia, from https://en.wikipedia.org/wiki/Penrose_tiling)

It is proved that the ratio of the number of the "thick" rhombi (Figure 2a) to the number of the "thin" rhombi (Figure 2b) in Penrose's tiling (Figure 3) strives to the golden ratio in the limit that means that the Penrose's tiles are based on the "golden" paradigm of the ancient Greeks. Note that the Penrose tiling [53] is of extremely important for crystallography.

Quasicrystals. On November 12, 1984, in the small article, published in the prestigious journal "Physical Review Letters", the experimental proof of the existence of a metal alloy with exceptional properties has been presented. Israeli physicist Dan Shechtman was the author of this experimental discovery. This alloy has shown all indications of a crystal. Its diffraction pattern was made up of bright and regularly spaced points, just like a crystal. However, this picture has been characterized by the presence of "*icosahedral*" or "*pentagonal*" symmetry, strictly forbidden in the crystal from geometrical considerations. Such unusual alloys were called *quasicrystals* [54].

Note that the *Penrose tiling* (Figure 3) is planar model of the *quasicrystals* and they had been used for theoretical justification of the *quasicrystals* [54]. This means that the *quasicrystals* are based on the "golden" paradigm of the ancient Greeks.

As highlighted in Gratia's article [55], the notion of the quasicrystal "*leads to expansion of crystallography, we only begin to explore the newly discovered wealth of quasi-crystals. Its importance in the world of minerals can be put on a par with the addition of the concept of irrational numbers to rational in mathematics.*"

In 2011 the author of this discovery, the Israeli physicist Dan Shechtman, was awarded by Nobel Prize in Chemistry, that is, the "golden" paradigm of the ancient Greeks has been recognized by the Nobel Committee.

Fullerenes. The fullerenes [56] are another outstanding contemporary scientific discovery, which has a relation to Platonic solids and consequently, to the "golden" paradigm of the ancient Greeks. This discovery was made in 1985 by Robert F. Curl, Harold W. Kroto and Richard E. Smalley. The title of fullerenes refers to the carbon molecules of the type C_{60}, C_{70}, C_{76}, C_{84}, in which all atoms are placed on a spherical or spheroid surface. In these molecules, the atoms of carbon are located at the vertexes of regular hexagons and pentagons that cover the surface of sphere or spheroid. The molecule of the carbon C_{60} (Figure 4a), called buckminsterfullerene, plays a special role amongst fullerenes. It is based on the so-called truncated icosahedron (Figure 4b) and has the highest symmetry.

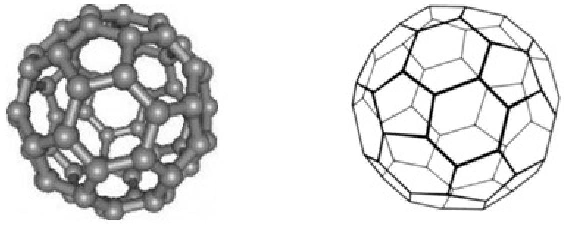

Figure 4. The molecule of the carbon C_{60} (a) and truncated icosahedron (b)
(Wikipedia, the free encyclopedia, from https://en.wikipedia.org/?title=Fullerene and
https://en.wikipedia.org/wiki/Truncated_icosahedron)

Fullerenes (Nobel Prize in Chemistry-1996) and quasicrystals (Nobel Prize in Chemistry-2011) are are the epitome of the "golden" paradigm of the ancient Greeks in modern science.

Fullerenes in the galaxy as the experimental confirmation of the self-organization and harmony of the Universe. The article [72], published in the Journal *Nature*, contains sensational information. The article presents an experimental proof of the fact that the *Milky Way* and other galaxies contain a large amount of fullerenes. Scientists say that fullerenes are actually contained in the galaxies in large amount what enshrines for them the status of "*nanofactories*;" so astronomers named them in 2011.

Commenting on this article, we can say that it contains very valuable information, which confirms the correctness of "harmonic ideas" by *Pythagoras, Plato, Euclid, Kepler, Klein*, who predicted the outstanding role of regular polyhedra ("Platonic solids") in the structures of science and Nature many millennia and centuries ago.

Due this experimental discovery, the fullerenes, based on the truncated icosahedron (Figure 4b) and the golden ratio, acquired the status of the main symbol of the "Universe Harmony." Thanks to the Nobel Prizes for fullerenes (1996) and quasicrystals (2011), theoretical natural sciences made great strides toward the "harmonic ideas" of Pythagoras, Plato and Euclid.

Icosahedron as the main geometrical object of mathematics: In the late 19th century, the prominent German mathematician Felix Klein drew a great attention on the Platonic Solids. He predicted outstanding role of them, in particular, of the icosahedron, in the future development of science and mathematics. In 1884 Felix Klein had published the book "Lectures on the Icosahedron" [71], dedicated to the geometric theory of the icosahedron.

According to Klein, the mathematical theories extend widely and freely in mathematics, like sheets of fabric, and they are united by some geometric objects what provides their broader and more general

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understanding. In Klein's opinion the icosahedron is precisely just such a mathematical object: "*Each unique geometrical object is somehow or other connected to the properties of the regular icosahedron*" [71]. In fact, Klein considers the icosahedron as such geometric object, which unites five branches of mathematics: *geometry, Galois' theory, group theory, invariant theory and differential equations*. Thus, Klein, by following to Pythagoras, Plato, Euclid, and Johannes Kepler, pays special attention to the fundamental role of the Platonic Solids for the development of science and mathematics.

Klein's ideas are entirely consistent with Proclus' hypothesis and are the epitome of the "golden" paradigm of ancient Greeks in mathematics and geometry.

Unfortunately, Klein's contemporaries could not understand and appreciate the revolutionary importance of Klein's ideas. However, their significance was appreciated one century later in 1982 when the Israeli scientist Dan Shechtman discovered a special alloy called *quasicrystal*. And in 1985 the researchers Robert F. Curl, Harold W. Kroto and Richard E. Smalley discovered a special class of carbon, named fullerenes. It is important to emphasize that *quasycrystals* are based on *Plato's icosahedron* and the *fullerenes* on the *Archimedean truncated icosahedron*. This means that in the 19th century Felix Klein made a brilliant prediction of the "golden" paradigm of the ancient Greeks in modern theoretical natural sciences.

Fibonacci numbers theory as the epitome of the "golden" paradigm of ancient Greeks in modern mathematics

The modern theory of Fibonacci numbers begun from mathematical discoveries of two Franch mathematicians: François-Édouard-Anatole Lucas (1842 – 1891) and Jacques Philippe Marie Binet (1776-1856). Lucas introduced the concept of generalized Fibonacci numbers, which can be calculated by following the general recurrent relation:

$$G_n = G_{n-1} + G_{n-2} \tag{118}$$

The main numerical sequence, *Lucas numbers* L_n, introduced by Lucas, are based on the following recurrent relation:

$$L_n = L_{n-1} + L_{n-2} \tag{119}$$

at the initial seeds

$L_1=1, L_2=3$

However, the *Lucas sequences* [73], introduced by Lucas, are his main mathematical contribution into *Fibonacci numbers theory*.

Binet's formulas (20), (21) are Binet's main mathematical contribution into *Fibonacci numbers theory*.

Lucas & Binet's studies stimulated further research in this area of modern mathematics. In 1963 a group of U.S. mathematicians created the Fibonacci Association. In the same year, the Fibonacci Association began publication of "*The Fibonacci Quarterly.*" In 1984 they began holding regular international conferences focusing upon "*Fibonacci numbers and their applications.*" The Fibonacci Association played a significant role in stimulating future international research.

American mathematician Verner Emil Hoggatt (1921-1981), professor at San Jose State University, was one of the founders of the Fibonacci Association and the magazine "*The Fibonacci Quarterly.*"

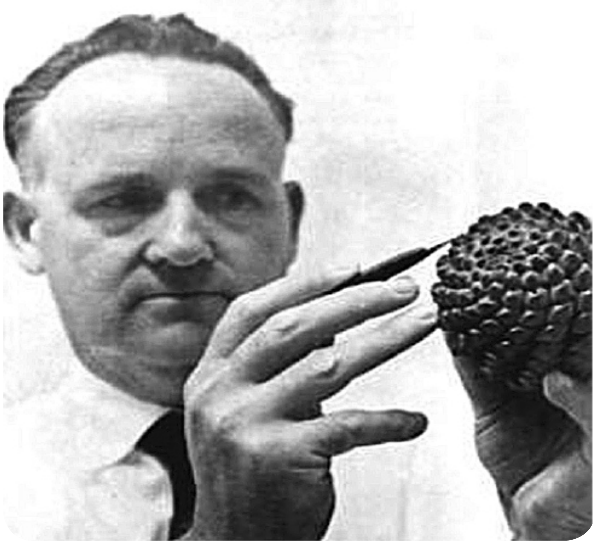

Figure 5: Verner Emil Hoggatt (1921-1981).

In 1969 Hoggatt published the book "*Fibonacci and Lucas Numbers*" [64], which is considered until now one of the best books in the field.

Brother Alfred Brousseau (1907-1988) was another prominent founder of the *Fibonacci Association* and *The Fibonacci Quarterly.*

Figure 6: Alfred Brousseau (1907-1988).

The Fibonacci Association has the rather unique and singular purpose of studying only the Fibonacci sequence. This raises some questions:

1. Why were the members of the Fibonacci Association and many "mathematics lovers" so focused on Fibonacci numbers?

2. What united these two very different people, mathematician Hoggatt and spiritual Brother Brousseau, in their quest to create the Fibonacci Association and establish the The Fibonacci Quarterly?

In an attempt to answer these questions regarding Hoggatt's and Brousseau's narrow focus on Fibonacci numbers, we need to examine

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some of their documents, in particular, their photographs, as well as the books and articles published in "*The Fibonacci Quarterly*."

In 1969, TIME magazine published an article titled "*The Fibonacci Numbers*" which was dedicated to the Fibonacci Association. This article contained a photo of Brousseau with a cactus in his hands. The cactus is of course one of the most characteristic examples of a Fibonacci phyllotaxis object. The article referred to other natural forms involving Fibonacci numbers. For example, Fibonacci numbers are found in the spiral formations of sunflowers, pine cones, branching patterns of trees, and leaf arrangement (or phyllotaxis) on the branches of trees, etc.

Alfred Brousseau recommended to the "lovers" of Fibonacci numbers "*pay attention to the search for aesthetic satisfaction in them.* ***There is some kind of mystical connection between these numbers and the Universe***."

However, Hoggatt holds a pine cone in his hands in photo (Figure 5). The pine cone, of course, is another well-known example of Fibonacci numbers found in Nature. From this comparison it may be reasonable to assume that Hoggatt, like Brousseau, believed in a mystical connection between Fibonacci numbers and the Universe. In our opinion, this belief may unite Hoggatt and Brousseau as a primary motivating factor in their work on Fibonacci numbers.

As indicated previously, Fibonacci numbers are connected with the *golden ratio*, since the ratio of two adjacent Fibonacci numbers strives to attain the *golden ratio* in the limit. This means that the Fibonacci numbers, approximating the golden ratio, are expressing the harmony of the Universe, because "*there is some kind of mystical connection between these numbers and the Universe*" (Alfred Brousseau). This means that *Fibonacci numbers are the epitome of the "golden" paradigm of the ancient Greeks in modern science* and *Fibonacci number theory is reflection of the golden*" paradigm of the ancient Greeks in modern mathematics.

Instead Conclusions: a role of the books "The Mathematics of Harmony" and "The "Golden" Non-Euclidean Geometry" in the development of the "golden" paradigm of modern science

In 2009 the International Publishing House "World Scientific" has published the book by Alexey Stakhov "The Mathematics of Harmony. From Euclid to Contemporary Mathematics and Computer Science" [2]. This book goes back in its sources to Euclid's Elements and is the first attempt to revive in modern science and mathematics the Pythagorean MATEM of harmonics, which lost in mathematics in the process of its historical development. The book by Alexey Stakhov and Samuil Aranson "The "Golden" Non-Euclidean Geometry. Hilbert's Fourth Problem, "Golden" Dynamical Systems, and the Fine Structure Constant" (World Scientific, 2016) [26] is continuation of Stakhov's book [2] and is the best application of the Mathematics of Harmony to Non-Eclidean geometry.

Introduction to the book "The "Golden" Non-Euclidean Geometry": The primary purpose of this book is to present in a concise manner the fundamentals of "*golden" non-Euclidean geometry* (hyperbolic and spherical), following from authors' original solution to Hilbert's Fourth Problem, the "*golden" qualitative theory of differential equations and dynamical systems*, and new theory of fine-structure constant, relating to Physics MILLENNIUM PROBLEM. It is intended to be accessible to a wider audience, including not only mathematicians, but also representatives of other areas of the theoretical natural sciences, including physics, chemistry, biology, botany, and genetics.

The book consists of 21 chapters.

Chapter 1 "*The Golden Ratio, Fibonacci Numbers, and the 'Golden' Hyperbolic Fibonacci and Lucas functions*" is a compact introduction to the theory of "Golden" hyperbolic Fibonacci and Lucas functions, based on the "golden ratio" and Fibonacci numbers, and their applications in the natural world ("Bodnar's Geometry").

Chapter 2 "*Fibonacci λ-numbers, 'Metallic Proportions,' and the Harmonic Hyperbolic Fibonacci and Lucas λ-functions*" is a short introduction to the theory of the Fibonacci λ-numbers, the "metallic proportions," and the Harmonic Hyperbolic Fibonacci and Lucas λ-functions. These are generalizations of the classic Fibonacci numbers and "golden ratio," and the "Golden" Hyperbolic Fibonacci and Lucas Functions. Stakhov's Harmonic Hyperbolic Fibonacci and Lucas λ-functions are the primary results of Chapter 2.

Chapter 3 "*Hyperbolic and Spherical Solutions of Hilbert's Fourth Problem*" is a concise introduction to Stakhov and Aranson's two fundamentally new mathematical results. Following an analysis of David Hilbert's famous article "Mathematical problems" and his approach to mathematical solutions ("Hilbert philosophy"), we discuss a solution to Hilbert's Fourth Problem based on hyperbolic Fibonacci λ-functions (introduced in Chapter 2). Next we discuss our solution to Hilbert's Fourth Problem based upon *spherical Fibonacci λ-functions*. Our results introduce a new class of elementary functions and demonstrate the existence of fundamental relations between hyperbolic and spherical Fibonacci functions which are presented in the comparative table ….

Chapter 4 "*Introduction to the 'Golden' Qualitative Theory of Dynamical Systems Based on the Mathematics of Harmony*" unites the Mathematics of Harmony by Stakhov … with the qualitative theory of differential equations and dynamical systems as developed by Aranson in his Doctoral dissertation (1990) and outlined in several mathematical works ….

Chapter 5 "*The Fine-Structure Constant as the Physical-Mathematical MILLENNIUM PROBLEM*" presents for the first time an original solution to the Physical MILLENNIUM PROBLEM, formulated in 2000 by David Gross, Nobel Prize Laureate in Physics (2004).

General Conclusions to the book "The "Golden" Non-Euclidean Geometry"

1. Discussing the history of mathematics and the development of new mathematical ideas and theories, we should draw particular attention to the central role of Euclid's Elements. Academician Andrey Kolmogorov identifies several stages in the development of mathematics. According to Kolmogorov, the modern period in mathematics began in the 19th century. According to him, "*expanding the content of mathematics became the most significant feature of 19th century mathematics*. At the same time, according to Kolmogorov, the creation of Lobachevsky's "imaginary geometry" became "*a remarkable example of the theory that has arisen as a result of the internal development of mathematics.... It is the example of geometry, which overcame a belief in the permanence of axioms, as a consecrated MILLENNIUM development of mathematics, and comprehended the possibility of creating significant new mathematical theories.*" As we know,

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"hyperbolic geometry" in its origins dates back to Euclid's 5th postulate. For several centuries, from Ptolemy to Proclus, mathematicians tried to prove this postulate. An initial brilliant solution to this problem was given by Russian mathematician Nikolai Lobachevsky during the first half of the 19th century. This marked the beginning of the contemporary stage in the development of mathematics.

2. At the interface of the 19th and 20th centuries, the eminent mathematician David Hilbert formulated 23 mathematical problems, which greatly stimulated the development of mathematics throughout the 20th century. One of these (Hilbert's Fourth Problem) refers directly to Non-Euclidean geometry. Hilbert presented mathematicians with the following fundamental problem: "*The more general question now arises: whether from other suggestive standpoints geometries may not be devised which, with equal right, stand next to Euclidean geometry.*" Hilbert's quote contains the formulation of a very important scientific problem, which is of fundamental importance not only for mathematics, but also for all theoretical natural sciences: *are there Non-Euclidean geometries, which are close to Euclidean geometry and are interesting from "other suggestive standpoints?*" If we consider it in the context of the theoretical natural sciences, then Hilbert's Fourth Problem is about searching for NEW NON-EUCLIDEAN WORLDS OF NATURE, which are close to Euclidean geometry and reflect some properties of Nature's structures and phenomena. Unfortunately, the efforts of mathematicians to solve Hilbert's Fourth Problem have not been very successful. In modern mathematics there is no consensus on the solution to this problem. In the mathematical literature Hilbert's Fourth Problem is considered to have been formulated in a ***very vague*** manner, making its final solution extremely difficult. As noted in Wikipedia, "*the original statement of Hilbert, however, has also been judged too vague to admit a definitive answer.*"

3. Besides the 5th Parallel Postulate, Euclid's Elements contain another fundamental idea that permeates the entire history of science. This is the idea of Universal *Harmony,* which in ancient Greece was associated with the golden ratio and Platonic solids. Proclus' hypothesis, formulated in the 5th century by the Greek philosopher and mathematician Proclus Diadochus (412 – 485), contains an unexpected view of Euclid's *Elements*. According to Proclus, Euclid's main goal was to build a complete theory of the regular polyhedra ("Platonic solids"). This theory was outlined by Euclid in Book XIII, that is, in the concluding book of the *Elements* which in itself indirectly confirms "Proclus' hypothesis." To solve this problem, Euclid included all the necessary mathematical information in the *Elements.* He then used this information to solve the main problem - the creation of a complete theory of the Platonic solids. The most curious (yet tell-tale) thing is that he had introduced the golden ratio early on in Book II for its later use in the creation of the geometric theory of the dodecahedron.

4. Starting from Euclid, the golden ratio and Platonic solids run like a "red thread" throughout the history of mathematics and the natural sciences. In modern science, Platonic solids have become a source for significant scientific discoveries, particularly of fullerenes (Nobel Prize in chemistry, 1996) and quasi-crystals (Nobel Prize in chemistry, 2011). The publication of Stakhov's 2009 book *The Mathematics of Harmony. From Euclid to Contemporary Mathematics and Computer Science* [2] is a reflection of this very important trend in the development of modern science (including mathematics): the revival of the "harmonic ideas" of Pythagoras, Plato and Euclid.

5. Argentinian mathematician Vera W. de Spinadel's metallic proportions [6], which are a generalization of the classic golden ratio, are a new class of mathematical constants of fundamental theoretical and practical importance. Besides Spinadel, several researchers from various countries and continents (e.g. French mathematician Midhat Gazale [7], American mathematician Jay Kappraff, Russian engineer Alexander Tatarenko, Armenian philosopher and physicist Hrant Arakelyan, Russian researcher Victor Shenyagin, Ukrainian physicist Nikolai Kosinov, Spanish mathematicians Falcon Sergio and Plaza Angel) also independently developed de Spinadel works [6]. All of this confirms the fact that the appearance of new (harmonic) mathematical constants has been maturing in modern mathematics.

6. In this book we have applied the "Mathematics of Harmony" [2] and Spinadel's metallic proportions [6] to the qualitative theory of dynamical systems. Of greatest interest here is the connection of Anosov's automorphism with de Spinadel's metallic proportions [6]. We outlined the prospects of the applications of this approach in such important areas as the qualitative theory of dynamical systems as applied to: small denominators, Reeb foliations, Pfaff's equations, etc.

7. The new classes of hyperbolic functions, based on the golden ratio and Fibonacci numbers (Fibonacci hyperbolic functions) (see Stakhov&Tkachenko's and Stakhov & Rozin's article [41], have become one of the most important results of the Mathematics of Harmony [2], having direct relevance to hyperbolic geometry. The λ-Fibonacci hyperbolic functions, based on the Spinadel's "metallic proportions" became an important step in the derivation of a general theory of "harmonic" hyperbolic functions [2]. It is important to emphasize here that in contrast with classic hyperbolic functions, the λ- Fibonacci hyperbolic functions, based on the golden and metallic proportions, have recursive properties, because they are fundamentally connected with Fibonacci numbers and their generalizations – λ-Fibonacci numbers. This fundamental mathematical fact allows us to place the λ- Fibonacci hyperbolic functions into a new and unique class of hyperbolic functions called recursive hyperbolic functions.

8. The research of Ukrainian architect Oleg Bodnar was a significant step in the development of hyperbolic geometry. He showed in [24] that a special kind of hyperbolic geometry, based on "golden" hyperbolic functions, has a wide distribution in Nature's flora and fauna, and underlies the botanic phenomenon of phyllotaxis (pine cones, cacti, pineapples, sunflower heads, etc.). His discovery demonstrates that hyperbolic geometry is much more common in Nature than previously thought. Perhaps Nature itself is the epitome of Bodnar's geometry. It is important to emphasize that Bodnar's geometry is based on the "golden" recursive hyperbolic functions, the base of which is the golden ratio.

9. From this point of view, the original solution to Hilbert's Fourth Problem, based upon Stakhov's Mathematics of Harmony [2], in particular on the metallic proportions [6] and λ-Fibonacci hyperbolic functions [44], is of special significance for mathematics and all the sciences. This is one of the main results

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of this book along with the previous authors' works in this area. These works demonstrate that there are an infinite number of new hyperbolic geometries, which "with equal right, stand next to Euclidean geometry" (David Hilbert). It is important to emphasize that the recursive Fibonacci hyperbolic functions, underlying our original solution to Hilbert's Fourth Problem, led to the creation of a new class of hyperbolic geometries, which have the common title **The "Golden" Non-Eclidean Geometry**.

10. This solution to Hilbert's Fourth Problem places the search for new ("harmonic") worlds of Nature in the center of the natural sciences (physics, chemistry, biology, genetics and so on). In this regard, we should draw special attention to the fact that the "silver" recursive hyperbolic geometry, based on the "silver" proportion $\Phi_2=1+\sqrt{2}\approx 2.41$, is the closest to Lobachevsky's geometry, based on the classical hyperbolic functions with base $e \approx 2.71$. Its distance to Lobachevsky's geometry is equal to $\overline{\rho}_{12}\approx 0.1677$, which is the smallest amongst all the distances for Lobachevsky's metric forms. We may predict that the "silver" the "silver" recursive hyperbolic geometry, based on the "silver" hyperbolic functions with the base $\Phi_2=1+\sqrt{2} \approx 2.41$, will soon be found in Nature and they along with Bodnar's geometry, based upon the "golden" recursive hyperbolic functions with the base $\Phi=\frac{1+\sqrt{5}}{2}\approx 1.618$, will become the most important recursive hyperbolic geometries of the physical and biological worlds of Nature.

11. We have attempted to extend the approach, used in [2], into the realm of spherical geometry. To solve this problem a new class of elementary functions has been introduced called λ- Fibonacci spherical functions having recursive properties. This class of functions allows for the solution to Hilbert's Fourth Problem with respect to recursive spherical geometry.

12. The subject of this book is of considerable interest from the standpoint of the history of mathematics and the prospects for its further development in close association with the theoretical natural sciences. This study unites the ancient golden ratio, described in Euclid's *Elements*, with both spherical geometry and Lobachevsky's hyperbolic geometry. This unexpected union led to our original solution of Hilbert's Fourth Problem and to the creation of a new class of Non-Euclidean geometry called recursive non-Euclidean geometry.

13. Viewing recursive non-Euclidean geometry and Bodnar's geometry through Dirac's Principle of Mathematical Beauty, we believe that if David Hilbert were alive today, he would have given his preference to solving the Fourth Problem in terms of the Mathematics of Harmony [2]. This "harmonic" solution unites *Leibniz's* "pre-established harmony" with Dirac's Principle of Mathematical Beauty as the foundation of a physical theory. This new solution to Hilbert's Fourth Problem and recursive non-Euclidean geometries can give vast opportunities for natural sciences in discovering applications of new non-Euclidean geometries in Nature.

14. After an original solution to Hilbert's Fourth Problem, we then focsed our attention towards Millennium Problems. Amongst these was a formidable challenge to determine whether any of physics' dimensionless constants are calculable. The main dimensionless constant that underlies the whole physical world is the fine structure constant. Through employing the Mathematics of Harmony, recursive Fibonacci hyperbolic functions and "golden" matrices we derived the Fibonacci Special Theory of Relativity, connecting the fine structure constant with time. The 2004 quasar observations from the Paranal Observatory in Chile are confirmation that the fine structure constant was different 10 billion years ago. Employing our theory we were able to match the value observed at Paranal to the 9th decimal place. The close approximation of our mathematical theory to the astronomical observational data confirms that our theory is valid. This original solution can be considered to be one of the most important theoretical results ever obtained through the intersection of mathematical theory and experimental physics. Thus providing an answer to Gross's Millennium Problem.

15. The fine-strcture constant determines the majority of dimensionless physical constants. The ratio M/m of the proton mass M to the electon mass m is one of them. By using the dependence of the fine-structure constant on time $T\alpha=\alpha(T)$, we have proved that the ratio $M/m(\alpha)$ also varies depending on the age of the Universe T.

Competing Interests

The authors declare that they have no competing interests.

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